VOUSSOIR ARCHES

APPLIED TO

STONE BRIDGES, TUNNELS, DOMES AND GROINED ARCHES.

BY

WM. CAIN, C. E.,
Carolina Military Institute, Charlotte, N. C.

REPRINTED FROM VAN NOSTRAND'S MAGAZINE.

NEW YORK:
D. VAN NOSTRAND, PUBLISHER,
23 MURRAY AND 27 WARREN STREET.
1879.

PREFACE.

THE following pages contain a discussion of the theory affecting the stability and strength of stone or brick arches, including bridges, culvert and tunnel arches, groined and cloistered arches, together with various styles of domes.

In the former treatise by the author, on *Voussoir Arches*, the theory of Dr. Scheffler (given in his German treatise on arches) for incompressible voussoirs was given, and applied to the experiments on arches, also to the case of a segmental stone bridge, the compressibility of the material being included in an empirical manner, which seemed to be justified in part from the consideration of the numerous experiments given.

In the present treatise the aim has been to make an advance in the theory, by considering the effect of the compressi-

bility of the material in a theoretical manner. This has the effect of modifying the assumption that the principle of the least resistance applies, except at the limit of stability, which last was shown by the experiments. It is gratifying to note, however, that the present theory conducts to nearly the same solution for the arch eccentrically loaded as that proposed in the former volume, so that the conclusion reached in both cases is the same. The connection between voussoir and solid arches will be given subsequently in treating the latter subject.

In the sections relating to curves of pressure corresponding to the maximum and minimum of the thrust, and parts of the treatment of underground arches and domes, with certain examples, so far as they refer to an imaginary incompressible material, the writer is again indebted to Scheffler's "Theorie Der Gewölbe. In the treatment of spherical and conical domes, assistance has been derived in part from Prof. Eddy's "New Constructions in Graphical Statics." The aim

throughout has been to carefully analyze hypotheses in the light of facts, and to present the subject in a simple manner, though not lacking in completeness; to which end the graphical method has been mostly employed.

THE AUTHOR.

VOUSSOIR ARCHES

APPLIED TO

STONE BRIDGES, TUNNELS, DOMES, ETC.

---

1. THE following is a continuation of a paper entitled "A Practical Theory of Voussoir Arches," which appeared in VAN NOSTRAND'S MAGAZINE for October and November, 1874, and afterwards reprinted as No. 12 of *Van Nostrand's Science Series*.

In that treatise, the principles affecting the stability of arches, upon the hypothesis of incompressible voussoirs, were exposed and applied to the investigation of numerous experiments upon wooden arches, *at the limit of stability*, with which they were found to agree. A segmental stone viaduct was likewise examined: the theory for incompressible voussoirs being modified *empirically* for the elastic materials used in construction.

It was mentioned in the former treatise, that we shall hereafter designate as Part I, that if the effects of the elasticity, causing the deformation of the compressible arch were known, that the investigation of its stability for a statical load could be effected.

The attempt is made, in what follows, to throw some light upon this effect of the compressibility of the voussoirs; and the empiricism, before mentioned, will be criticized in the light of the deductions, as well as from a further consideration of the experiments themselves.

Afterwards, the precise part played by the spandrels will be pointed out and certain theories concerning them discussed. The subject of the theory of arches will then be extended to *underground arches*, *groined* and *cloistered arches*, and *domes*; and practical examples will be given, worked out in detail, to illustrate the investigation—as far as it can be made—of the stability and strength of such structures.

EFFECT OF THE ELASTICITY OF THE MATERIALS.

2. We shall introduce the present subject with some comments on the fourth experiment given in Part I. Fig. 1 represents one-half of a wooden gothic arch and pier of fourteen inches span; the depth of voussoirs being two inches, the horizontal width of pier, 1.9; its height, ten; and the uniform thickness of arch and pier, 3.65 inches. The contour curves of each half arch are described from the opposite springing points. The voussoirs were constructed of equal weight, the pier weighing 2.3 voussoirs. The inner edge of top of pier coincides with the intrados at the springing. With no weight on the crown the arch and pier stood, but fell with a slight jarring, such as a person walking across the room. Now, as the crown joint and joint 5 opened on the intradosal, and joint 3 on the extradosal side, even when the arch stood; the voussoirs bearing at the very edges opposite the opening; it is evident that the arch, at the moment

of rupture, was slightly deformed; *i. e.*, did not have exactly the figure above.

If that deformation had been noted and the figure drawn to correspond, then the resultants of the pressures on joints 1, 3 and 5 would have passed almost through the very edges; but assuming that the arch at the instant before rupture had the figure above, we find, if the horizontal thrust Q at the crown acts 0''.1 below the summit, that with the value of Q as given, the resultant pressure at joint 3 passes 0''.1 from the intrados, and at joint 5, 0''.2 from the extrados.

To pass a curve of pressure through the points noted on joints 0 and 3, lay off on the direction of Q prolonged, the distances $\overline{c1}$, $\overline{c2}$,... to the verticals through the centres of gravity of the loads from the crown resting on joints 1, 2, . . . respectively. In this case, as detailed in Part I, the distances $\overline{c1}$, $\overline{c2}$, . . . are respectively 1.7, 3.19, 4.39, 5.26 and 6.24 inches; the weights resting on joints 1, 2, 3, 4 and 5 being respectively, 1, 2, 3, 4 and 6.3 voussoirs.

These weights are laid off in order on the force line $\overline{05}$ on the right, and a line drawn from 3 parallel to the line 33 in the figure of the arch, cutting off, $Q=0.63$ voussoirs by scale. Then if from the points 1, 2, 3 . . . on Q prolonged, we draw lines parallel to the oblique lines through 1, 2, 3, . . . of the force diagram, to the corresponding joints, we find the *centres of pressure* on the joints, the broken line connecting which is called the *line of pressures.* The magnitude and direction of the resultants are given by the oblique lines of the force diagram.

3. Now, if the material of which the arch is composed was absolutely incompressible, *i. e.*, able to sustain any finite effort on a mathematical line, then the centers of pressure at joints 0 and 3 would have passed through the very edges, and the arch would have balanced on a higher pier. We propose to show that this deformation of the arch was due entirely to the compressibility of the material. Again, it is a most important problem to ascertain the distribution of

the molecular stresses on any joint, having given their resultant there in position, direction and magnitude. This latter subject is not restricted in its applications to arches alone, but applies to any plane joint or supposed section in a solid on which the resultant of all the external forces acting on the structure can be found; as on any joint of a chimney, retaining wall, abutment, arch, etc.

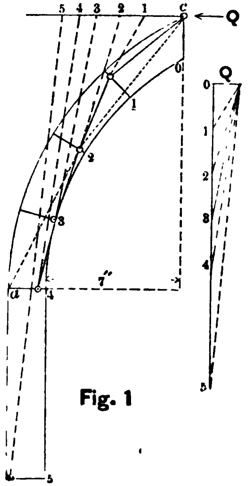
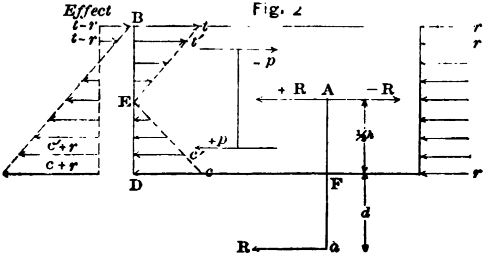


Fig. 1

Referring to Fig. 1, we see that the resultants on the joints are not generally perpendicular to them: resolve them into components ⊥ and ∥ to the joints to which they correspond. The latter, or "shearing force," is resisted by the friction of the joint, resisting sliding, it is only the former component that we shall consider.

4. Conceive a slice DDA, Figs. 2 and 3, of width unity; the cross-section BD, perpendicular to the plane of the paper, being a *rectangle.*

Fig. 2



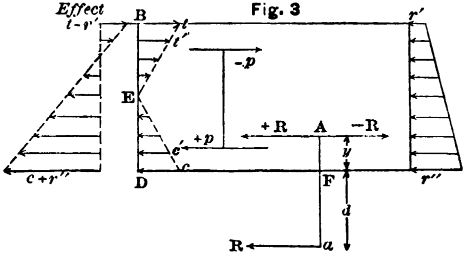
Also suppose the resultant R of all outward forces on the cross-section BD, to lie in a plane parallel to the paper, through the center of the slice and per-

pendicular to the supposed cross-section. Let us suppose *first* that the cross-section can develop both *tensile and compressive* resistances as in a solid beam, and that R passes through *a*, *without* the cross-section.

If at some point, A in the medial plane, we conceive two opposed forces +R, −R, parallel and equal to R; the force R with the force−R forms a right-handed couple RR, that can be replaced by the equal couple $\overline{pp}$ or the forces, *t*, *t'* . . . . *c'*, *c*, equal and opposed to the uniformly increasing tensile *resistances* from E to B and the compressive *resistances* from E to D, E lying in the center of gravity of the cross-section.

5. In Fig. 3 A lies anywhere, in the

Fig. 3



medial plane, *within* the cross-section (a further limit will presently be indicated); in Fig. 2, A lies in the center of gravity of the cross-section, so that $+R$ passes through E.

The force $+R$ at A, Fig. 2, is decomposed into forces, r, r . . . ., *supposed* uniformly distributed. In Fig. 3, $+R$ at A is *supposed* decomposed into r', ..., r'', straight lines, limiting the arrows representing the forces.

Since the final "effect" is to give forces $t-r,'$ . . . . , $c+r,''$ (Fig. 3) limited by *straight* lines, the result must be the same as in Fig. 2, since the forces $t-r$, . . . $c+r$, as limited by straight lines, can have but one disposition in order that R at a may be their resultant.

This would not be so, if some curved line limited the ordinates $r,'$ . . . . r''; which decomposition is thus incorrect if we assume, as is usual in the flexure of beams, that the forces exerted by the fibres and their consequent compressions or extensions, *within the limits of elasticity*, are directly as their distance from

the neutral axis, or point of no strain, shown in the "effect" diagram of both figures.

6. PROP.—*If the joint* BD *is a plane joint;* i.e., *can offer compressive, but no tensile resistances, then if* R *falls inside of the joint, anywhere between* A *and* F *Fig.* 2, *it is decomposed into compressive forces only, which decrease regularly from the edge* D, *nearest* R, *towards the other edge, and thus, are proportional to the ordinates of a trapezoid or triangle.*

This is easily proved, if we assume, that *when* one edge of a plane joint is more compressed than the other, the actual shortening of the fibres and hence, (within the limits of elasticity) the forces acting on them, must be directly as their distance from the neutral axis.

There are only three suppositions:

1st. Suppose an *equal* shortening of the fibres on the whole joint. The stresses per square unit are thus the same throughout the whole extent of the joint; but then R, lying on one side of the center, cannot be their resultant.

One edge then must be compressed more than the other.

2d. But the actual shortening of the fibers cannot be greatest at the edge farthest from R, for then, by hypothesis the stresses must regularly increase in going towards the edge farthest from R; but then R cannot be their resultant. This second supposition is then false.

3d. These two suppositions proving incorrect, the third as stated in the proposition is correct.

R can be, and is, the resultant of the forces distributed according to the law of the trapezoid; the most compressed edge lying nearest R.

7. When R lies so near the edge, that the limit of elasticity is passed in the case of some of the fibers, then although it looks probable that the actual shortening of the fibers is directly as their distance from the neutral axis, yet the corresponding resistances are no longer proportional to the compressions, for those fibers whose limit of elasticity is passed; hence R is no longer decomposed

according to the ordinates of a trapezoid, except on a portion of the joint, the resistances being less than by this law as we approach the most compressed edge, for the balance of the joint.

8. Poncelet asserts that the "law of the decomposition of molecular forces at the exterior surface of a solid body" has not been solved hitherto. If we restrain ourselves to practical cases, such as that of one arch stone pressing upon another, or any single block pressing upon another block throughout its whole extent, the above *is* a solution; for in practice we should not allow the resultant to approach so near the edge that the limit of elasticity of any of the fibers is passed; and within this limit the solution is founded upon the same hypothesis as that used in discussing the laws of flexure.

It would seem as useless as impracticable to solve a problem, such as presented by Poncelet, of finding, the distribution of the pressure due to a heavy elastic prism, resting on a non-deformable horizontal plane; or as suggested by another author, of considering the stresses caused by an isolated weight resting on a common pedestal, &c., &c.

9. Where one block rests upon more than one, the decomposition becomes

indeterminate. Thus, suppose at the foundation course of a pier, abutment, retaining wall, &c., the resultant passed outside of the middle third. We shall see presently, that if the course resting on the foundation is of one block, then the joint will open at the edge farthest from the resultant.

Where the stones are not cemented together firmly, it is doubtful if the resultant is decomposed according to the law of the trapezoid, where the courses are formed of many stones. The middle third is recommended as a good practical limit however. Still it must not be thought that it is a cure for all evils. Whenever the resultant on any course does not coincide with the center of figure, there will be settling on the side towards it; so that no pier etc., can be regarded as non-deformable; and the amount of this yielding to allow is simply a practical question; a slight opening of the joints, if not seen, being of itself of no matter, unless the pressures are thereby increased too much for safety, or

water is permitted to enter, or some practical objection, other than want of required stability, is experienced.

10. Refering again to Fig. 2, and calling f the strain t or c, we know from the theory of flexure that

$$\text{Moment of R at } a\text{, about } A = R(d+\tfrac{1}{2}h) = \tfrac{1}{6}fh^2,$$

whence,

$$f = \frac{6R(d+\frac{1}{2}h)}{h^2} = t = c \qquad (1)$$

$$\text{The uniform compression } r = \frac{R}{h}$$

$$\therefore (t-r) = \frac{R}{h^2}(6d+2h), \ldots\ldots \qquad (2)$$

$$c+r = \frac{R}{h^2}(6d+4h) \ldots\ldots\ldots \qquad (3)$$

11. Supposing tensile resistances at the joint, these formulæ give correct results for the solid beam; and likewise for a plane joint, when R is so near A, that $t-r$, $t'-r$, . . . , may all act as compressive forces; since, by this decomposition, the law of the trapezoid previously established, art. 6, holds; R being the

resultant of the regularly increasing stresses.

12. Now let $t-r=0$. From eq. 2, $d=-\frac{1}{3}h$ or $a\text{A}=\frac{1}{6}h$. As long as $a\text{A}$ is less than $\frac{1}{6}h$, there are only compressive resistances on the joint; but if the resultant leaves "the middle third" of the joint BD, then $(t-r)$ is positive and we *assume tensile resistances* at B to oppose the forces.

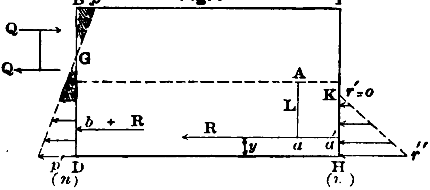
13. *But* suppose the joint BD can offer no tensile resistances, then it is wrong to decompose R as before, if by another method of decomposition stability is assumed. Many writers seemed to have overlooked the method of decomposition proved in art. 6.

Thus, R at a, Fig. 4, is the resultant in position on the joint BD. We may decompose R according to the ordinates of a triangle as at (m), if we assume the law of the trapezoid as before, r' is now 0, hence $y=\frac{k}{3}$, where $k=\text{KH}=$height of trapezoid or triangle.

Placing $\overline{Aa}=L$, we have,

$$HK=h=3\ \overline{Ha'}=3(\tfrac{1}{2}h-L) \quad . \ (4)$$

Fig. 4



14. Let us now compare this result with the case shown at (n), where the beam is supposed to supply tensile resistances along BG. Call the resistances per square unit at B and D, p and p' respectively; *i.e.*, $p=t-r$ and $p'=c+r$.

From similar triangles, calling $DG=x$,

$$p' : x :: p : h-x \therefore x=\frac{p'\,h}{p+p'},$$

Now, replacing $(d+\frac{1}{2}h)$ in eq. (1) by L; deducing p and p' as in (2) and (3), and substituting, we have,

$$x=DG=\frac{\frac{R}{h}\left(\frac{6L}{h}+1\right)h}{\frac{R}{h}\ 2\frac{6L}{h}}=\frac{h^2}{12L}+\frac{h}{2} \quad . \ (5)$$

Now from eqs. (4) and (5)

For $L=\frac{1}{6}h$,	$HK= h$,	$DG= h$,
$\frac{1}{3}h$,	$\frac{1}{2}h$,	$\frac{3}{4}h$,
$\frac{5}{12}h$,	$\frac{1}{4}h$,	$\frac{7}{10}h$,
$\frac{1}{2}h$,	0,	$\frac{2}{3}h$,

That is HK$<$DG when L lies between $\frac{1}{6}h$ to $\frac{1}{2}h$; or the point of no strain in the beam lies nearer the edge DH, where the joint can oppose no tensile resistances than when we suppose them exerted.

14. Let us now compare the strains p' and r''. Representing R by the area of the triangle whose base is KH and altitude r'', we have,

$$R=\frac{kr''}{2}=\frac{3yr''}{2}$$

$$\therefore r''=\frac{2}{3}\frac{R}{y} \quad . \quad . \quad . \quad (6)$$

Again moment of R at a about $A=R(\frac{1}{2}h-y)$ &c. See art. 10. Hence,

$$p'=(c+r)=\frac{R}{h^2}(4h-6y) \quad . \quad . \quad (7)$$

Thus for any values of y between 0 and $\frac{1}{3}h$ (see table) we find that, $r''>p'$, always :

y	r''	p'
$\frac{1}{12}h$	8	$3\frac{1}{2}$
$\frac{1}{6}h$	4	3
$\frac{1}{4}h$	$\frac{8}{3}$	$\frac{5}{2}$
$\frac{1}{3}h$	2	2
	$\frac{R}{h}$	$\frac{R}{h}$

There is therefore greater compression at edge DH when the real forces are as at (m) than when the beam can oppose tensile resistances as at (n). Also since KH $<$ DG, within the same limit ($y=o$ and $\frac{1}{3}h$) the compression is more uniformly distributed at (n) than at (m); hence the beam will bend more when, as in the voussoir arch, a joint BD can only oppose compressive resistances, than when, as in a solid beam, tensile resistances can be exerted. As a consequence of this compression at lower edge, there being none (according to our hypothesis—the law of the trapezoid) at K in case (m), the joint above K must open, and could in fact be removed without interfering with the distribution of the forces at all. . . . Some writers have erroneously asserted that the opening was due to tensile forces. It may be observed, that if, as in the voussoir arch, these tensile forces are unbalanced rupture must ensue.

16. *The decomposition of* R *cannot be the same for an open joint as for a solid beam, when* L *lies*

between $\frac{1}{6}h$ and $\frac{1}{2}h$; for if, for a solid beam, we assume the disposition (m), the edge nearest R is most compressed, and the joint above K would tend to open ; but as there are tensile resistances there that prevent it, the disposition (m) is not correct for the solid beam; hence some other disposition as (n) is correct.

17. It is evident, from the reasoning in art. 6 that, *in a voussoir arch, whenever the resultant on a joint does not pass through its center, that the edge nearest the resultant is most compressed, and the arch is consequently deformed.*

If some external force, as a spandrel thrust, keeps the line of pressures in the middle of the arch ring, then there will be no deformation of the arch, save that due to the uniform compression at each joint.

Conversely, if the arch retains its shape, save that due to a uniform compression at each joint, the line of pressures must coincide with the center line of the arch ring; for otherwise there would necessarily be deformation of the arch, which is not supposed.

18. It is evident, that as the resultant

pressures on the joints are farther removed from the center, the deformation is greater; increasing gradually until, when the line of pressures leaves the middle third of the arch ring, the joint just begins to open, this opening increasing, [see Fig. 4 (m)] until when the resultant is very near the edge, the strain r'' on the most compressed fiber exceeds the ultimate strength of the material and crushing ensues, followed or accompanied by rotation. For very light arches this crushing may not be perceptible. Thus, in the experiment given Fig. 1, there was deformation of the arch, before rotating began, due to the compressibility of the material, joints 0, 1 and 5 compressing most at their outer, joints 2, 3 and 4 at their inner edges. The resultants at joints 0, 3 and 5, obeying the law of nature's economy of force, pass nearly through the very edges; so that $k=\overline{\mathrm{KH}}=3y$ (Fig. 4, m) is very small, and as the face of one voussoir can be supposed to rotate about K, the opening at I must be very appre-

ciable, as it was at joints 0, 3 and 5 in the actual experiment.

19. It is likewise evident that the greater the compressibility of the material, the greater the deformation of the arch; hence, with weak materials, rotation will occur sooner, *i.e.*, with less loads or lower piers, than for less compressible materials. This principle is proved experimentally by exps. 13 and 14 of part 1, with cloth joints.

The above theory now perfectly explains, that apparent anamoly in the experiments, of the resultant on the base of a pier, made up of several bricks, approaching the center as the height of the pier was increased; the arch and pier being at the limit of stability. The true explanation is, that it is due to a compression of the edges, causing a deformation of the arch; so that if the exact figure of the arch at the instant of rotation could be obtained, we should find that the line of pressures approaches closely the very edges at the joints of rupture.

In fact this is necessarily so, for as

these edges alone bore, just before rotating, the line of pressures must pass through them. The same remarks apply to all the experiments. It was not attempted to find the figure of the deformed arch just before rotating, for the figure in most cases was not constant and hence impossible to obtain. In truth a very limited time scarcely permitted of the experiments that were performed.

20. It is well to note that so long as R remains within the *inner third* of the arch ring, that compressive forces act on the *whole extent* of the joint, and thus there will be no opening. Eq. (3) will give the unit strain on the fiber at the most compressed edge, noting that d is now minus. When $d=-\frac{1}{8}h$, $(c+r)=\frac{2R}{h}$, or double the compression $\frac{R}{h}$, if the resultant were uniformly distributed on the joint.

Hence at those joints where the resultant cuts the joint $\frac{1}{8}$ the depth of joint, from either edge the strain induced in the most compressed fiber, is double that

due to a uniform distribution of the resultant on the joint.

21. *It is clear likewise, that inversely, if in any voussoir arch the joints are all closed, that the line of pressures keeps somewhere within the middle third;* for if it did not, then there would be compression, uniformly increasing, as at Fig. 4 (m) over *only a part* of the joint; so that while the fiber at K is unaltered, those fibers on the part KH are compressed, which necessitates the opening of the joint above K; but this is against the supposition of closed joints, hence the actual line of pressures keeps within the middle third. Eq. (6) gives the value, $r'' = \frac{2}{3}\frac{R}{y}$, of the unit strain on the edge nearest R, when R lies in the *outer* third of the arch ring, the mortar being supposed to offer no appreciable tensile resistance.

Thus if R passes $\frac{1}{4}$ depth joint from an edge, $r^{4} = \frac{2}{3}\frac{4R}{h} = \frac{8}{3}\frac{R}{h}$, being slightly over

double the unit strain $\frac{R}{h}$, if R was uniformly distributed.

22. For arches of medium spans, it can matter little, so far as strength and stability are concerned, whether, by the aid of a spandrel thrust or other device, or from inherent strength, the line of pressures is restrained to the middle third or to limits, $\frac{1}{4}$ depth joint from edges, or other limits. Thus in the stone viaduct of 50 feet span, given in Part I, the horizontal thrust at the crown, in a slice 1 foot wide, is nearly 25 tons, the depth of voussoirs being 2.5 feet. Now if the compression were uniform, the strain per square foot would be 10 tons.

If the resultant at the crown passed, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$ depth joint from extrados or intrados, the strains per square foot at the most compressed edge would be 20, 26.7, and 33.3 tons respectively. Since the crushing weight of granite varies from 400 to 800, of limestone from 250 to 600, and of sandstone from 200 to 300 tons per square foot, it is evident there can be

no danger of crushing in these cases for such materials. The crushing weight of best brick in cement, Trautwine gives as 50 to 70 tons per square foot; of brick alone, at from 50 to 300.

If the material, in the most compressed edge, is not to be subjected to more than $\frac{1}{5}$th of its breaking weight, then brick should not be used of the proportions above.

23. If an iron band is placed around that part of the extrados, where the joints would otherwise open, it may entirely or partially prevent this opening. The arch then ceases to be strictly a voussoir arch, so that this device will not be further noticed.

24. We have been careful to expose this law of the decomposition of the resultant on a cross section in detail, since so many writers have fallen into error on this point: some applying the principles affecting a solid beam to the open joint, and thus discovering a veritable case of rotation when the line of pressures passes outside the "middle third" of the

arch ring, due to supposed tensile forces; and others again rejecting the law of the trapezoid entirely, and falling back upon the theory of incompressible voussoirs.

It is needless to give all the various theories, even in outline, that have been proposed by so many able writers. Suffice it to say, that it is probably agreed, that the true solution of the arch is intimately connected with the law of its compressibility; or more plainly, the law of its deformation due to its elasticity.

The aim is, therefore, whilst not attempting a thorough solution, to endeavor to present clearly some important points bearing on this subject.

25. It is assumed as proved, that if, *in any completed arch, no joints open, that the actual line of pressures keeps within the middle third of the arch ring.* In fact, it is well to design the arch ring so as to satisfy this condition.

It does not follow, by any means, that if this actual line is found outside the middle third, that the arch will fail, as

Rankine, *e.g.*, asserts. We have just seen that both strength and stability may be satisfied when this line approaches the edges at certain joints quite closely, so that other limits might, in many cases, with safety be instituted.

But it seems advisable, entirely from a practical point of view, not to have any open joints; it gives an appearance of insecurity and besides may leave too small a margin for shocks.

This principle does not apply to culverts, buried out of sight and never sustaining any shocks or much variation of pressure; hence it is extravagance to give them the same depth of keystone as a stone bridge. It is for the same reason proper to increase the depth of arch stones for a railroad arch bridge over the sizes usually employed in road bridges, as the live loads are heavier and move faster. For tunnel arches the judgment of the engineer must be largely exercised in allowing for the different thrusts of the various materials found in tunneling; and here it is better to be on the safe

side and use a deep arch ring to allow for variations in thrust, especially if the soil is treacherous.

26. Let us recur again to the gothic arch, Fig. 1; which, as remarked, spread outwards about the haunches when set up on the solid piers. If with the hands the tops of the piers are moved inwards until the span is just 14 inches, the joints are all closed as stated in Part I, p. 50. The line of pressures is, therefore, confined to the middle third. If the top of the pier is pushed still farther in, the crown joint opens at the top, bearing at the bottom edge; and this is so, whether the other joints that may open are wedged up just to close or not.

Now there are some intermediate positions for the top of the pier when the horizontal thrust at the crown acts at different points along the joint: for as the top of pier is moved gradually inwards from its limiting position the thrust at the crown must travel *gradually*, down the joint, until it reaches the lower edge, when, of course, the lower edge alone bears.

It cannot jump at once from its highest to its lowest position.

This granted, a very important proposition is established: *that by cutting or fitting the arch stones in a certain manner that the line of pressures at the crown may be made to pass through any desired point of the crown joint.* Thus, in this case, with a span less than 14 inches, the thrust at the crown may be confined to any point of the crown joint between the top of the middle third limit and the bottom of the joint; and the voussoirs may be so cut that no other joint opens, even if the crown joint opens.

The same remarks apply to any other joint. A line of pressures may thus be made to pass at will through any point at the crown joint, and through some point below the previous one, generally on any other joint. Thus the arch may be pivoted at the crown and at the haunches, as see exps. 3 and 10 of Part I; or the line of pressures may be compelled to take the same position by properly chipping away part of certain joints; so that

the true line of pressures is, after all, dependent on the mason. If he so cuts and lays the stones that on completion the bridge shows no open joints, as is the rule, then the line of pressure is somewhere within the inner third.

27. Next, suppose the piers removed and that the arch stands upon a firm support, the joints being closed as stated: where is the actual line of pressures? The line as first drawn corresponds to the principle of least resistance; but this cannot be the true one, *since, for no other reason, the consequent rotation about the upper edge of the crown joint and the lower edge of joints* 3 *or* 2, *would necessitate a rotation about the extradosal edge of joint* 4. But this last cannot occur, unless the line of pressures passes nearly through *a*, the outer edge of joint 4; involving a new curve of pressures, as shown by the dotted lines, c_2a, similar to the *actual* one drawn in Fig. 11 of Part I, referring to the first experiment, in which the joints opened as just described. In fact when the crown is lowered, the

haunches must spread, and consequently the springing joint be most compressed on the extrados side. Now, in practice, this spreading always occurs at the haunches, so that the line of pressures there is below the center line, whilst at the top and springing it is above it or outside of it.

If no joints open, as was the case, the line must keep within the inner third besides. Now, if the law of this deformation of the arch was known, the curve could be located; as it is, we can only approxmate to its true position, by noticing the manner in which arches settle, or fail as just shown. Now, it was proved by numerous experiments in Part I, *that when an arch failed by rotation, in every case the line of pressures corresponded to both the maximum and the minimum of the thrust.*

This is illustrated by figures 11, 12, 13, 23, and 24, of Part I, which figures refer to a few of the experiments made on the wooden arches. Now, it will probably be admitted that, with voussoirs that fit

perfectly before decentering, and with incompressible abutments, that the deformation of the arch due to its compressibility, is in the same direction as that when the arch is at the limit of stability from a similar kind of loading; *i. e.*, the curve of pressures crosses the centre line the same number of times, and near the same places. Then it seems highly probable, under the conditions assumed, that *the actual line of pressures,* in such an arch, *is confined within such limiting curves, approximately equidistant from the center line of the arch ring, that only one curve of pressures can be drawn therein, corresponding, therefore, to the maximum and minimum of the thrust in the limits taken.*

Assuming this to be true, let us critisise the constructions given for the segmental bridge in Part I. Thus, in Figs. 9 and 10, the line of pressures is nearer the centre line than drawn. The curve for the limit of stability is represented in Fig. 13, when the weight is at the crown ; and by Fig. 24, for a single weight on one haunch.

Now, according to the principles exposed above, a line of pressures for the bridge, Fig.

21, loaded eccentrically, should be drawn through the lower middle third limit, at joint 6, on the loaded side, the upper limit, at joint 6, on the unloaded side, and a point at the crown joint slightly lower than before. On a large scale drawing (3 feet to the inch), assuming the thrust at the crown joint to act 1.1 ft above the intrados, we find that the line of pressures passes above the middle third limits at joint 2, under the load 0.2 ft.—its max. departure—just touching the lower limit at joints 1 and 2, on the unloaded side. If preferred, the direction and amount of the thrust at the crown to pass through the given points on joints, 6, may be found by successive trials in place of computing them. Thus, assume the direction of the thrust: then find its amount to pass through the point on one joint 6, and use this amount in finding the center of pressure on the other springing joint, which should coincide with the point taken on that joint, otherwise, try again. Three trials only sufficed in this case.

From the above, we see that the depth of arch ring should be increased 0.2, in order that a line of pressure, giving the max. and min. thrust in the limits, may everywhere keep within the middle third. The recommendation, though, is repeated to increase the depth of the arch ring 0.5 foot, to allow for the influences mentioned in the next article.

If the object, therefore, is simply to

investigate the stability of a proposed structure (which is indeed the real object of our investigation), it is evident that, *if any line of pressures can be inscribed within limits, so that no crushing occurs, that the arch is stable against rotation for statical loads.* For with less horizontal thrusts than that taken, the curve departs more and more from that which corresponds to the ultimate maximum and minimum, without which the arch cannot fail by rotation. As explained above, it is well to confine this line of pressures to the *middle third* of the arch ring, so that no joints open. If the line drawn corresponds to the minimum, but not to the maximum of the thrust within the limits taken, it is not the actual line of pressures (if the abutments are firm, etc.), since this is probably contained within still narrower limits. In fact there will be an excess of stability in this case. In arts. 55, *et seq.*, the characteristics of the maximum and minimum thrusts will both be investigated in the most general manner.

Remark.—It should be observed that *if*, in a voussoir arch, there are no mortar joints, and the stones are cut so perfectly that the compression is the same next the joints, as in the body of the stones, then *when* the pressure line keeps within the inner third, the conditions are exactly similar to the case of the solid or rigid arch "fixed at the ends." For the graphical treatment of this case, the reader is referred to Professor Greene's articles on this subject in *Engineering News* for 1877, in connection with Bell's article in VAN NOSTRANDS' MAGAZINE, Vol. 8; also to Professor Eddy's "New Constructions in Graphical Statics." The analytical treatment is given in full in Du Bois' "Graphical Statics." On testing the curve of pressures corresponding to the max. and min. of the thrust, of the segmental bridge with the eccentric load examined in Part I, by the three conditions, $\Sigma M=o, \Sigma My=o$, and $\Sigma Mx=o$, the center line of arch ring having been divided into 16 equal parts, it is found that the curve of pressures should be raised slightly, to satisfy these conditions, which are simply, that the tangents at the springing are fixed, the span is invariable, and the vertical displacement of one springing above the other, equals zero. It seems, however, useless to enter into this refinement for actual bridges with mortar joints, rough beds, etc.; therefore, it is not mentioned further.

28. The abutments or piers have previously been considered as unyielding. If their tops lean outward, from a yielding of the foundation on the outer side where the greatest pressure is ordinarily thrown, or if the span lengthens from the compression caused, we see from the remarks on Fig. 1, that the actual line of pressures corresponds to a less horizontal thrust, so that it approaches the intrados at the springing, and, of course, retreats farther within the abutment at its base. If the abutment continues to yield, it approaches still more the contour curves at the crown and haunches, so that a sufficient yielding may cause the line of pressures to approach very nearly the very edges at those joints.

If the abutment is narrow, and especially if each course is built of a single block, like the voussoirs, for the thickness taken, it becomes really a continuation of the arch, so that all the principles of art. 27 apply; the base of the abutment becoming the actual springing joint.

Now, in practice, the abutments do yield somewhat; so that in most arches the actual line of pressures corresponds to a less thrust than as suggested in the previous article. In view of the fact that the arch-stones may not fit perfectly (for openings of the joints do occur sometimes, slate rock being driven in them), in addition to the foregoing, it seems impossible to say exactly where the true line of pressures is to be found. In most bridges the joints do not open, so that in them, it is confined to the middle third.

To be on the side of safety in designing piers or abutments, the maximum thrust in the limits may be used. If, however, the top of the pier leans, the thrust forcing it over becomes less, whilst the opposing thrust of the arch on the other side increases. When there are a series of arches resting on piers, the resultant on them due to dead load is vertical if the arches are all alike, or have the same horizontal thrust. In all cases, the pier should be investigated

when one span is loaded and the adjacent one unloaded, exactly as shown in Part I, Fig. 22.

When the piers are very high, a series of arches are often placed below the first, which must, therefore, supply sufficient horizontal thrust to keep the line of pressures within the middle third of the horizontal joints of the piers from the lower arches to the base. The part of the piers between the series of arches are to be examined as before. Some magnificent structures have thus been built with tiers of arches. Instance of modern construction, the Morlaix Viaduct; also, the bridge over the Seine at the Pont du Jour, Paris. Often the piers of the upper series rest upon the arches beneath, which are generally of greater span than the upper series.

29. In addition to the influences exerted on the line of pressures by the cutting of the stones, a similar one producing similar effects is *Temperature.* Stoney says, "With increased temperature the crown rises and joints in the parapets

open over the crown, while others over the springing close up. The reverse takes place in cold weather; the crown descends, joints over the springing open and those over the crown close. When stone or iron arches are of large span, these movements, from changes of temperature, will generally dislocate to a certain degree the flagging and pavement of the roadway above. This is very conspicuous in Southwark Bridge."

These combined influences, like the shocks due to rolling loads, can only be guarded against by empirically increasing the depth of the arch ring over that due to the statical loads, as recommended in art. 27.

Practical Conclusions.—In view of the combined influences of misfits, shocks, temperature, and yielding of abutments the true curve of pressures may be found in wider limits than those in which it has the characteristics of the max. and min. of the thrust in the arch ring.

If we design an arch ring then, in which a line of pressures corresponding

to the max. and min. of the thrust can *just* be drawn in the middle third, some joints may still open from some of the influences mentioned above; therefore it is recommended to use somewhat narrower limits than the middle third. Or better, require that a line of pressures may always be inscribed in the inner third, and increase the depth of arch ring empirically to allow for the other influences; since they are different for each bridge; dependent as they are upon the thrust, height of pier or abutment, kind of foundation, accuracy with which the stones are cut and other considerations detailed above.

INFLUENCE OF THE SPANDRELS.

30. The part played by the spandrels has not previously been enquired into. When the centers of an arch are struck, it is found, particularly for full center arches, that the crown descends and the haunches spread outwards. This spreading is *resisted* by the spandrels, which if solidly built, exert horizontal forces suffi-

cient to prevent much lateral motion of the arch and thus help to keep the joints closed. If there is no tendency to spread in the arch, the spandrel exerts no horizontal resistance; it being simply a wall resting on a rigid arch ring, which thus cannot differ in its action from any other fixed foundation of the same shape. As generally built, with vertical joints next the arch, and even with inclined joints there, in most cases, there is no tendency to *slide*, which alone can cause an *active* horizontal thrust.

One author, M. Y. Von Villarceau, assumes that the spandrels exert an *active* horizontal thrust, proportioned, like liquids, to the depth below the surface! He thus obtains, after hundreds of pages of intricate calculations, "*the hydrostatic arch*"—one of the worst forms of arch a constructor could well choose. What a regret that such industry could not have been devoted to a more practical end.

Rankine's theory will be noted further on.

If in any of the experiments given in

Part I, especially with the gothic arch where the spreading was very appreciable, a sufficient horizontal thrust was exerted by the hands or spandrel walls against the extrados, the arches, even under much greater loads would have been stable. Similarly in a stone bridge.

If the spandrel thrust, at different points was given, the investigation of the stability of the arch ring would be as simple as that for a culvert or tunnel arch given further on.

But it seems impossible to estimate it.

If each spandrel course was composed of one stone, and the yielding of these stones noted, then by the law, "*ut tensio sic vis*," we could deduce the horizontal resistance offered by each course. But if each course was made up of several stones, the forces corresponding to the compressions of the mortar joints would be too uncertain to rely upon, and thus the horizontal resistances could not be computed.

31. As, however, arches, such as the semi-circular, semi-elliptical, etc., that

require a very effective spandrel thrust to maintain stability, are often built, it is necessary that the engineer comprehends all the reactions experienced in such arches, although he may not be able to precisely estimate them in tons, etc.

To this end, let us consider the semicircular arch, Fig. 5, of 100 feet span, and 3 feet depth of keystone; the solid spandrel extending to 3 feet above the top of the keystone.

Divide up the spandrel by vertical lines, 5′ apart for 40′ from the crown, than 2′ apart for the next, 10′ and 1′ apart for the remaining 3 feet. The joints 1, 2, 3, ... are then drawn as in the figure.

Our object is first to ascertain the weight from the crown resting on any joint, and the position of the centre of gravity of this weight. The line of pressures, disregarding the spandrel thrust is then drawn as explained for Fig. 1.

In the following table the first column indicates the joint. In the next four

Joint.	*w*	*v*	*s*	*c*
	5.	3.1	15.5	2.5
1	4.9	3.	14.7	2.5
	5.	3.5	17.5	7.5
2	4.9	3.	14.7	7.4
	5.	4.5	22.5	12.5
3	5.	3.	15.	12.3
	5.	6.	30.	17.5
4	5.2	3.	15.6	17.1
	5.	8.	40.	22.5
5	5.3	3.	15.9	22.
	5.	10.7	53.5	27.5
6	5.75	3.	17.2	26.8
	5.	14.2	71.	32.5
7	6.2	3.	18.6	31.7
	5.	18.6	93.	37.5
8	6.87	3.	20.6	36.6
	2.	22.3	44.6	41.
9	3.1	3.	9.3	39.9
	2.	25.	50.	43.
10	3.3	3.	9.9	41.9
	2.	27.9	55.8	45.
11	3.7	3.	11.1	43.8
	2.	31.4	62.8	47.
12	4.3	3.	12.9	45.8
	2.	35.9	71 8	49.
13	5.13	3.	15.4	47.7
	1.	39.9	39.9	50.5
14	3.23	3.	9.7	49.1
	1.	43.5	43.5	51.5
15	4.2	3.	12.6	50.2
	1.	49.	49.	52.5
16	10.	3	30.	51.3
			1003.6	

m	S	M	C
39.			
37.	30.2	76	2.5
131.			
109.	62.4	316	5.1
281.			
184.	99.9	781	7.8
525.			
267.	145.5	1573	10.8
900.			
350.	201.4	2823	14.
1471.			
461.	272.1	4755	17.4
2307.			
589.	361.7	7651	21.1
3488.			
754.	475.3	11893	25.
1829.			
371.	529.2	14093	26.6
2150.			
415.	589.1	16658	28.3
2511.			
486.	656.	19655	30.
2952.			
591.	731.7	23198	31.7
3518.			
735.	818.9	27451	33.5
2015.			
476.	868.5	29942	34.5
2240.			
632.	924.6	32814	35.5
2572.			
1539	1003.6	36925	36.8
36925			

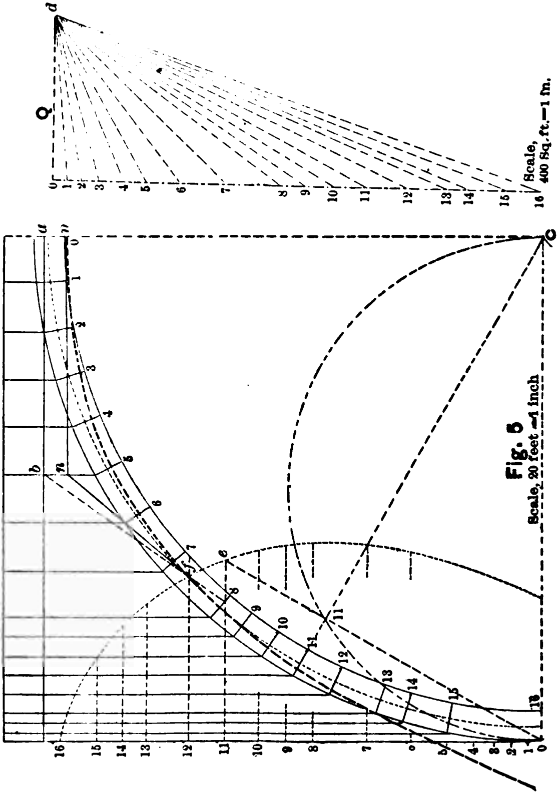


Fig. 5
Scale, 20 feet = 1 inch

columns, the upper numbers, opposite any joint number, refer to the trapezoidal figures; the lower numbers to the voussoirs on which the trapezoids rest:

We assume that the surface s of a *trapezoid* is equal to its horizontal width, $w \times$ mean height v; the latter being measured approximately from the top of the spandrel to the extrados along the medial vertical line.

For the *voussoirs*, surface$=s=w\times v=$ length measured along center line$\times$ depth (3′ in this case), see Part I, p. 78. Column c gives the horizontal distances from the crown to the medial vertical of the trapezoid (assumed to pass through its center of gravity), and to the center of gravity of the voussoir respectively. The product, $s.\ c=m=$moment of s about the crown. Column S now gives the area of the trapezoids and voussoirs from the crown to any particular joint, and is formed by the successive addition of the numbers in column s. Similarly M is formed from m. The quotient $\frac{\text{M}}{\text{S}}=\text{C}=$horizontal distance from

the crown to the center of gravity of the surface S corresponding to the same joint. Thus, for joint 8, the sum of the moments of each trapezoid and voussoir from the crown to joint 8=M= 11893; the sum of their area=S=475.3; hence by mechanics $C=\frac{M}{S}=25$. If we take a slice of the arch of the width unity, then S will represent the corresponding volumes, and is proportional to their weights.

The whole surface of the arch and spandrel = rectangle − quadrant = $56\times53-\pi\frac{2500}{4}=1004.5$, differing 0.9 square foot from the value found approximately in the table.

32. Let us now pass a curve of pressures through a point *a*, 1′ below the extrados at the crown, and a point 1′ from the intrados along joint 8.

To do this we lay off on the horizontal through the upper point, $\overline{ab}=C=25'$ to the left of the crown. From the point *b* so determined, draw a straight line to the lower point at joint 8. This line

gives the resultant on joint 8 in position and direction. Next, laying off to scale in the force diagram to the right, the surfaces S in order, and drawing through 8 a line ‖ the direction of the resultant just found, it will cut off on a horizontal through o the value $\overline{od}=Q$ of the horizontal thrust at the crown.

Drawing lines through 1, 2, . . . (force diagram) to d so found, we have the directions and magnitudes of the resultant on joints 1, 2, . . . represented by their lengths. To find the center of pressure on any joint as *e. g.*, 5, we lay off $C=14$ on the horizontal $\overline{ab}$. From the point thus found draw a line ‖ $\overline{d5}$ of force diagram; where it intersects joint 5 is the center of pressure on that joint: for the thrust Q at a, combined with the weight resting on joint 5, act ing at its center of gravity must give the resultant on joint 5 in position, magnitude and direction. Similarly for other joints.

The broken line traced through these centers of pressure—not drawn in the

figure to avoid confusion—is the "*line of pressures.*" In this case it everywhere keeps within the middle third of the arch ring, except at joints 15 and 16. At joint 15 it passes in the arch ring 0.6 feet from the extrados; at joint 16, 1.9 feet outside of the extrados. [If there was no spandrel, the true line of pressures must lie in the arch ring, if possible, and satisfy the conditions of art. 27.]

33. At joint 7, the center of pressure is at the middle of the joint; the line of pressures then goes below the center line of the arch ring, approaches nearest the intrados at joint 8; at joint 13 it again crosses the center line, and keeps above it to the abutment. Joints 8, 9 and 10 are the most compressed on the intrados side. Joint 8 is often called the joint of rupture.

34. Now if this were the true curve of pressures, the effect of the compression, not being uniform on the joints, would be to lower the crown and spread outwards the haunches (see art. 28). So that if the spandrels were solidly built

up to joint 7, they would offer horizontal resistances, in addition to their vertically acting weights to partially prevent the deformation of the arch ring.

Let us suppose, for an instant, that the spandrels are absolutely incompressible; the arch, then, cannot change shape where the tendency to spread occurs; therefore, the compression must be uniform on such joints; *whence the actual line of pressures must pass through the centers of those joints.*

This compression along the arch ring shortens it slightly, but we shall neglect this shortening. Our untrue hypothesis conducts us to the following construction, similar to the one given in Rankine's Civil Engineering, article 138: lay off, on the extreme left vertical, the loads from *o* (Fig. 5) upwards. Now, *assume* that the curve drawn tangent to the resultants on the joints coincides with the "line of pressures" (which hypothesis is sensibly incorrect as we approach the springing). Draw $\overline{oe}$ parallel to the center line at joint 11—the con-

struction shown effects this easily, by drawing $\overline{o,11}$ through the intersection of $\overline{c,11}$ with the semi-circle. Next draw a horizontal line through 11 on the scale of loads, to the intersection e with the line just drawn; then $\overline{oe}$ represents the magnitude and direction of the resultant at joint 11, whose two components $\overline{o,11}$ and $\overline{e,11}$ are respectively, the load from the crown to joint 11, and the *total horizontal thrust exerted below* joint 11.

On repeating this construction for each joint, we find the horizontal thrust exerted below each joint; the horizontal thrust, then exerted upon a single voussoir, as that between joints 11 and 12, by the spandrels, is thus the difference between the line $\overline{e, 11}$ and the horizontal $\overline{f, 12}$. At joint 8, the horizontal thrust obtains its maximum; and above this point, in this case, the spandrel would have to exert tensile forces to cause the center line to become the true line of pressures; but as it cannot do this, the horizontal thrust from joint 8 to the crown is constant.

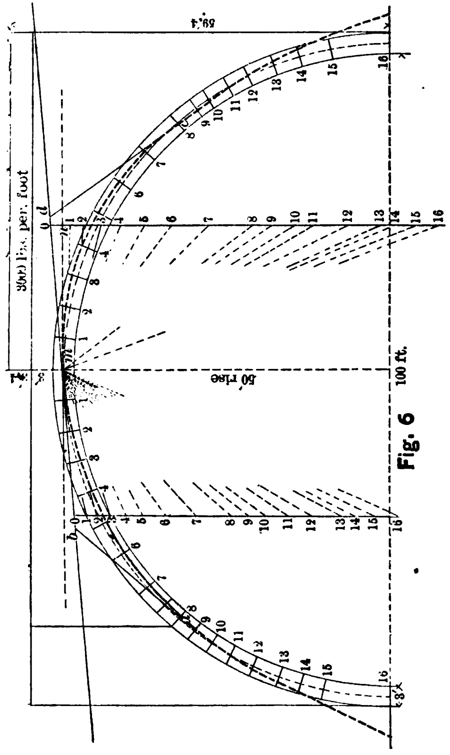


Fig. 6

35. If an accurate construction should be desired, to ascertain the horizontal force supplied by the spandrel at the extrados of each voussoir, in order that the center line may be the curve of pressures, from the joint where spreading first occurs to the abutment, we may proceed as follows: assume the thrust on that joint to pass through its center; then having assumed the position at the crown of the horizontal thrust, we find, as in art. 32, the magnitude and direction of the resultant on the joint considered. Combine this resultant with the weight of next voussoir and load, acting through their common center of gravity, and the resultant so found with such a horizontal force, acting through the middle, approximately of the extrados of the voussoir, as to cause the final resultant to pass through the center of the next joint. This construction may be continued to the abutment.

If a force polygon is drawn to one side, the amounts of the horizontal thrusts supplied to each voussoir by the spandrel becomes evident.

36. Continuing the construction of art. 34, we draw a tangent to the center line at joint 8, to intersection n with the vertical through the center of gravity of the load from the crown to joint 8, and from this point draw mn horizontal, to intersection m with the crown joint.

From *m* the center of pressure on the crown joint, the curve of pressures to joint 8 is drawn as before explained in art. 32. It is shown by the dotted line through *m* which point is 0.2 feet below the crown. The curve is continued below joint 8, supposing *no* spandrel thrust, and cuts the springing joint 4.5 feet to the left of the extrados.

On the supposition of incompressible spandrels however, the true curve is that drawn through *m* to joint 8; then it follows the center line. Rankine indeed, asserts that a linear arch parallel to the intrados and drawn within the middle third is the true curve, whence *n* and *m* may be slightly raised or lowered. (See Rankine's Civ. Eng., art. 285.)

This cannot be if the arch is to perfectly preserve its figure (art. 34). Again, for compressible spandrels, the line of pressures about joint 8 *must* lie below the center line, *never* above it (art. 28).

37. But is this the true curve? Decidedly not. If there were no spandrels, and the abutments yielded suffi-

ciently, the curve would be somewhat as drawn in art. 32, tending to flatten the arch from the crown to joint 7; then to render it more convex; and still lower, causing the greatest compression to occur at the extrados.

Now, *theoretically*, with spandrels built in the usual manner, when the centers are struck, the tendency of the arch to spread at the haunches *causes* the compression of the spandrels, which thus, in partially resisting this spreading, put forth horizontal forces. The true line of pressures then between joints 7 and 13, about, must pass *below* the center line; below joint 13 approximately, it keeps to the left of the center line. The latter follows from the fact that any spreading at the haunches is accompanied with a diminished compression of the intrados at joint 16, (see art. 27).

Again, the spandrel thrust must be greatest where the spreading is greatest; whence from joint 8 the spandrel thrust necessarily diminishes down to the springing or near it, where it ceases. No

spandrel thrust is experienced above joint 8 about, if the spandrel moves as one mass.

In practice, the spandrel is not solidly built above joint 8, except on the faces of the arch, so that on that account, the thrust above joint 8 will be small if any.

38. *Practically*, the first solution above conducts to this absurdity; the arch should be unstable because m lies below the arch ring, but the depth of keystone was obtained from a comparison of examples in actual practice, so that no engineer would believe that a full center arch of 100 feet span and 3 feet depth of keystone should be unstable. Thus, from Rankine's formula, founded entirely on practice, the depth of keystone should be, for a single arch $\sqrt{.12 \times r} = 2.45$ feet, or for an arch of a series $\sqrt{.17 \times r} = 2.9$ feet; r being the radius at the crown, 50 feet in this case.

Again, *in practice*, the crown falls on decentering; hence it seems probable that the true line of pressures there is *above* the center line, not below it.

39. *Now if the joints keep closed, the actual line of pressures probably passes above the center line at the crown; between the center line and the lower middle third limit at joint* 8 *about; from this point it again approaches the center line, crossing it about joint* 13 *and passing near the outer middle third limit at the springing joint, keeping throughout within the middle third of the arch ring. The spandrel then, from the point of greatest spreading to, or near, the abutment, must exert the least horizontal resistances that will effect this object. Some idea of their magnitudes could be gained from the construction of art.* 35, *if we knew two points, say at the crown and joint* 8, *through which to pass the curve of pressures, assuming its position below joint* 8.

If this be true, then the thrusts exerted by the spandrels are evidently much less than as given by Rankine's construction, and the spandrel in a semi-circular arch does not sustain the whole of the horizontal thrust. Rankine's theory is the

only one, except Y. Von Villerceaus, that has yet been proposed to evaluate the spandrels' influence.

40. If the top of the backing of an arch is sloped downwards from the arch, it may not be capable, near the top, of exerting much horizontal thrust.

If we *knew* the total spandrel thrust down to a certain horizontal joint, the weight of the spandrel above this joint multiplied by the coefficient of friction of stone on stone is the force that resists the sliding tendency; so that the height of spandrel, on this supposition, is easily computed.

As Rankine's construction gives an excess of spandrel thrust, it may be used to evaluate the least height of backing, both loose and solid, to be used.

41. Whatever doubt may exist as to the precise *measure* of the forces exerted by the spandrel, its important action in preventing deformation of the arch ring from a theoretical stand point is rendered plain by the above discussion; and the locus of the true curve of pressures is

more precisely ascertained than hitherto; which was the object to be accomplished in the present instance. It is usual with constructors to strike the centers, after the keystone is driven in and the backing carried up such a distance above the "joint of rupture" (as joint 8 is often called) that the arch ring will be stable when the supports are removed. This would seem to be a matter to be determined from practice; though a curve of pressures can be used in approximately testing the stability of the unfinished arch.

It may be observed that, if the resultant at joint 8 maintains its position and direction approximately, that the center of pressure at the crown for the completed arch will be lower than for the unfinished arch, *if* the vertical through *b* moves to the left as the arch is completed; otherwise the reverse happens.

42. *Width of Piers and Abutments.*—For full security, in these arches where the spandrels exert a marked influence, the horizontal thrust at the crown may

be taken as acting at the lower middle third limit, whence the center of pressure at the base of an abutment is determined exactly as in art. 2, Fig. 1, (also, see art. 28). The right half of the arch being supposed removed and Q being applied at the crown to produce the same effect; its combination with the weight of the semi-arch and abutment must give the resultant acting on the base of the latter, irrespective of internal actions, such as the real distribution of the spandrel, thrust, &c. For the piers, suppose again, for safety, that Q acts at the lower middle third limit at the crown; then find the resultants acting at the level of the springing, due to the arches on both sides of a pier; on combining them with the weight of pier, acting at its center of gravity, the center of pressure at the base can be found. It is a good practical rule to limit this center of pressure in both abutments and piers to the middle third of the base (see art. 9); and to cause it to approach the center as the foundation becomes more insecure.

43. Let us next suppose the full center, Fig. 6, loaded with cars weighing 3000 pounds per lineal foot from the crown to the right abutment. If this load bears upon a width of six feet, it is equivalent to a layer of stone of same density as the bridge (150 lbs. to the cubic foot), 3.4 feet high, as represented in the figure.

The following table* for the right half of the arch is made out exactly as explained in art. 31, only the voussoir numbers, for each joint, are placed above the corresponding spandrel numbers. Now let us pass a curve of pressures $\frac{1}{12}$th depth of arch ring below the center line at the crown joint and joints 8, through *m*, *a* and *c*. The thrust at the crown is generally inclined. Let us deduce some general formulæ to enable us to find it.

* See Table on pp. 70, 71.

44. Call P_1 = weight from crown to joint 8 on left,

a_1 = its lever arm about a,

P_2 = weight from crown to joint 8 on right,

a_2 = its lever arm about c,

Q = horizontal component of thrust at m,

b_1 = lever arm about a

b_2 = " " " c

P = vertical component of thrust at m. considered positive when it acts downwards as regards pressure from the right half of the arch upon the left.

g_1 and g_2 are the horizontals from a and c respectively to the vertical through the crown.

Now suppose the right half of the arch removed and its effect replaced by P and Q acting at m; we have taking moments about a.

$$a_1 P_1 + g_1 P = b_1 Q. \quad . \quad . \quad . \quad (5)$$

Next, conceive the left half removed, &c., and take moments about c.

$$a_2 P_2 - g_2 P = b_2 Q. \quad . \quad . \quad . \quad (6)$$

Eliminating Q, we have

$$P = \frac{a_2 b_1 P_2 - a_1 b_2 P_1}{g_1 b_2 + b_1 g_2} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (5)

$$Q = \frac{a_1 P_1 + g_1 P}{b_1} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

	s	c	m
	14.7		37
1	32.5	2.5	81
	14.7		109
2	34 5	7.5	259
	15.		184
3	39.5	12.5	494
	15.6		267
4	47.	17.5	822
	15.9		350
5	57.	22.5	1283
	17.2		461
6	70.5	27.5	1939
	18.6		589
7	88.	32.5	2860
	20.6		754
8	110.	37.5	4125
	9.3		371
9	51.4	41.	2107
	9.9		415
10	56.8	43.	2442
	11.1		486
11	62.6	45.	2817
	12.9		591
12	69.6	47.	3271
	15.4		735
13	78.6	49.	3851
	9.7		476
14	43 3	50.5	2187
	12.6		632
15	46.9	51.5	2415
	30.		1539
16	52.4	52 5	2751
	1183.8		41700

S	M	C
47.2	118	2.5
96.4	486	5.
150.9	1164	7.7
213.5	2253	10.5
286.4	3886	13.6
374.1	6286	16.8
480.7	9735	20.2
611.3	14614	23.9
672.	17092	25.4
738.7	19949	27.
812.4	23252	28.6
894.9	27114	30 3
988.9	31700	32.
1041.9	34363	33.
1101.4	37410	33.9
1183.8	41700	35.2

See more general formulæ in Part I, art. 12, and in art. 63 following.

45. From the tables of arts. 31 and 43, we have $P_1=475.3$, $P_2=611.3$; and their centers of gravity are distant from the crown, respectively, 25. and 23.9 feet.

From the drawing, we have thus, $a_1=13.7$, $a_2=14.8$, $b_1=b_2=17.5$, $g_1=g_2=38.7$, whence,

$$P=\frac{-a_1P_1+a_2P_2}{2g_1}$$

$$=\frac{-13.7\times 475.3+14.8\times 611.3}{2\times 38.7}=32.8$$

$$Q=\frac{a_1P_1+g_1P}{b_1}$$

$$=\frac{13.7\times 475.3+38.7\times 32.8}{17.5}=444.6$$

From m lay off to the right, horizontally, $Q=444.6=\overline{mn}$; then vertically upwards, $P=32.8=\overline{no}$: om represents the resultant at the crown joint. Now, lay off the force lines $o, \ldots.$ 16 from colums S in the tables; so that $\overline{m1}$, $\overline{m2}, \ldots$ now represent the directions and magnitudes of the resultants on joints 1,

2, ... right and left of the crown. Their positions are found as follows: draw a horizontal through *m*, and lay off on it, the numbers in column C; the first table referring to the left half of the arch, the last table to the right half.

From the points so found, draw vertical lines to intersection with $\overline{mo}$, produced if necessary; which thus give the points where the inclined thrust at *m* is to be combined with the weight from the crown to any joint, to find the resultant on that joint; whose intersection with it is thus the center of pressure for that joint.

Thus, P_1 acts 25′ to left of *m*: lay off 25′ on the horizontal through *m*, then drop a vertical to intersection *b* with *mo*; then draw $\overline{ba} \parallel m8$ of force line for left of arch, to find *a* the center of pressure for joint 8. Similarly *d* and *dc* are found for joint 8 on the right. These should be the first constructions made to test the values of P and Q found, which correspond to the line of pressures passing through *a*, *m* and *c*.

The line of pressures thus drawn passes below the middle third of the arch ring, on the unloaded side, the following amounts in feet: at joints 2, 3, 4, 5 and 6, .3, .4, .3, .2 and .1 respectively; it then crosses the arch ring, passes above the middle third about joint 12, and cuts the springing joint 4.5 feet outside of the arch ring.

On the loaded side it passes above the middle third 0.1 at joints 4 and 5; then across the center line and is just tangent to the lower middle third limit at joint 10, below which it again crosses the arch ring and passes into the abutment, cutting joint 16 about 3 feet outside of the arch ring.

46. On the unloaded side this curve below joint 8 follows very closely the curve drawn in Fig. 5; so that if horizontal forces, as large as the ones supposed supplied by the spandrels by Rankine's construction are applied here, the line of pressures will coincide nearly with the center line. On the loaded side the line of pressures touching the lower

middle third limits at joints 9 and 10 would be forced too far in by the supposed horizontal thrusts. As mentioned in art. 39, this spandrel thrust is probably much less than given by the above construction; thus if c is at the center about, and a at the lower limit, the thrust at the crown being raised slightly above the center, the spandrel thrusts, necessary to keep this line of pressures within the middle third say, are nearer the true ones than as given by the previous construction. It seems, at present, impossible to locate exactly the true curve of pressures when spandrel thrusts are exerted.

47. If, however, the abutments are large enough and unyielding, and the spandrels sufficiently strong to resist the thrust, the arch *cannot fail* by the sinking of the crown, which is the usual method of failure of full center arches. Thus Gauthey says (see extract in Haupt's Bridges, p. 126), "let cvc', Fig. 7, be the intrados of any arch, whether semicircular, elliptical, gothic or com-

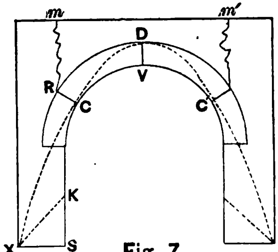


Fig. 7

posite. Let D be the crown of the extrados, or back of the arch, which is supposed to be filled up level with the haunches at *m* and *m'*. If a weight be placed upon the crown too great for it to bear, it yields, and the arch stones open beneath at the crown, while the extrados is found to open at some point on each side; either at the spring, if it be a flat arc of a circle, or about 30 degrees of a semicircle, or at various other points if it be composed of arcs of circles, tangent to each other, and of various rises, whether $\frac{1}{4}$ or $\frac{1}{3}$ or $\frac{1}{6}$ of the span; and the arch *only falls*" (italics our own) "by pushing aside the abutments at C and C′, the opening at R extending itself up to

the top at m and m'. It has, moreover, been observed that when the abutment gives way, it leaves a portion of itself standing, viz., XKS, the line XK being at an angle of 45° with the horizon, which only adheres by the strength of the mortar or cement made use of."

This last fact should be observed in designing pillars of any kind. The line of pressures for the kind of rotation just mentioned is shown by the dotted line.

If the arch ring has a very small depth some of the voussoirs may crush; or again, the arch may fail by rotation by the *crown rising* and the haunches falling in, as in the bridge over the Tâfe (see art. 51). Eccentric loads, as we saw in Part 1, cause the haunch under them to fall, and the opposite one to rise. In these cases the rising of the crown—we repeat, the only way in which a bridge can fail with solid, immovable abutments, and spandrels sufficient to resist the thrust against them—tends again to relieve the spandrels of a certain amount of strain, thus causing the curve of

pressure to *rise* at the crown, with a less horizontal thrust, and thus partially counteract the tendency towards rotation about the intrados at the crown.

The depth of voussoir, in this case, can only be determined by an empirical rule founded on practice, as given below. A rolling load evidently causes different spandrel resistances from the dead load, though Rankine's construction does not make any difference for the unloaded side; another proof of the incompleteness of his theory.

48. Let us investigate the part *amc* of the bridge as though it was a *segmental bridge*, resting upon fixed abutments at *a* and *c*. It is only needful to examine it for an eccentric load (see Part I.) Pass a curve of pressures through the upper middle third limit at joint 8 on the left, the lower limit at joint 8 on the right, and 1.25 ft. above the intrados at the crown joint. We find on a drawing of 3 ft. to the inch, $g_1=39.25$, $g_2=38.48$, $a_1=14.25$, $a_2=14.58$, $b_1=17.16$ and $b_2=17.81$. As before, $P_1=475.3$, $P_2=611.3$: whence, by eq. 7,

$P=23.8$, $Q=449.1$.

The line of pressures drawn with these values keeps everywhere within the middle third, barely touching the lower limit at joint 2 on the left, and passing 0.16 ft. inside of upper limit at joint 3 on the right, and corresponding (art. 27) nearly to the maximum and minimum of the thrust in the limits chosen.

The span of this arch is 75.45 feet, its rise being 17.2 feet, between $\frac{1}{4}$ and $\frac{1}{5}$ of the span. The deformation is small for this segmental bridge, so that the spandrel resistance may be neglected, or rather regarded as simply adding to the stability of the bridge, already safe, unless from the dynamical effects of moving loads. Concentrated loads on one side (as in art. 12, Part I.) should next be tried, in positions that cause the most hurtful effects. This influence, together with the influences of art. 28, may cause an increase of depth of arch ring of half a foot over the three feet.

49. Recurring to the semi-circular arch, Fig. 6, it is evident that for an unsym-

metrical load, the spandrel on the opposite side to the load will exercise a thrust for a greater height above the springing than for a uniform load, as compare the line of pressures in the two cases.

It may be asked, why does not the part of the arch below the joint of rupture, a, Fig. 6, act as a part of the abutment simply; so that if the part amc satisfies the conditions of stability, when it is treated as a segmental bridge, why should not the whole bridge be stable?

Rankine's empirical formulae for the depth of keystone, $\sqrt{cr}$, r being the radius of the arch, and c a constant, seems to be founded on such a hypothesis; for by it, the depth of keystone is the same for spans of any length, provided the radius is the same. Thus, if $r=50$ feet, as in Fig. 6, this depth is the same for any span between 100 and 0 feet!

The above query may be answered thus: the actual line of pressures in an arch bridge like Fig. 6, is dependent upon the form of the arch below the points a and c, since the deformation of

this part induces the spandrel thrusts (art. 37) which influences the position of the true line of pressures.

In fact, suppose that the line found in Fig. 6, on the supposition of no spandrel thrusts, to be the true one for a segmental bridge *amc* of span *ac:* by the reasoning of art. 46, the actual spandrel thrusts exerted below *c* would force this line out of the middle third, certainly, and most probably out of the arch ring, whereas, the spandrels are supposed to allow no joints to open at least. Trautwine's empirical formulæ for depth of key in feet, *d*, is,

$$d = \frac{\sqrt{r + \text{half span}}}{4} + .2;$$

and is more agreeable to theory than Rankine's; although for railroad bridges it gives too small values; at least for bridges of 50′ to 75′ span, and rises $\frac{1}{5}$ to $\frac{1}{6}$ span, as for the two segmental bridges examined in art. 48, and in Part I., art. 12. It is evident again, from the reasoning of art. 28, that it would be advisable

to increase the depth of arch ring, the smaller the ratio of width to height of abutments; and by the same rule, an arch in a series should have a greater depth of voussoirs, as recognized in Rankine's rule for that case.

50. Having resigned the above arch, that requires a spandrel thrust to keep it from falling, to the domain of empiricism, it may be asked, if by some device of construction, whether it may not be brought within the limits of a strict investigation? Plainly, if the depth of arch stones be increased towards the abutments, so that a line of pressures, with a constant horizontal thrust, can always be inscribed within the middle third, the arch will be stable; and the spreading will be so much diminished that the arch will require but little spandrel thrust to cause stability, so that its influence may be neglected in constructing the line of pressures.

This increase in the depth of arch stones is earnestly recommended; as well as the continuation of the arch ring into the abutment, when, as in segmental

bridges, there is some danger of sliding at the springing.

If necessary, the increase in depth of the arch ring may be made up of several stones. They should, of course, break joint with stones above and below them, and be well bonded with the spandrels.

HEIGHT OF SURCHARGE, IN ORDER THAT THE CENTER LINE OF ARCH RING MAY BE A POSSIBLE CURVE OF PRESSURES.

51. Let it be required to find the proper height from the soffit to the top of roadway, in order that the center line of the arch may be a possible line of pressures.

The black line at the top, Fig 8 shows the line of roadway for the segmental arch of 100′ radius and 4′ depth of keystone, found by the following approximate construction:

Divide the semi arch into nine portions aa_1, a_1a_2, . . . , each of the same horizontal length; the weight of each portion is nearly proportional to its medial vertical line, limited by the soffit and roadway

(yet to be found), drawn through the centers c_1, c_2, ... The horizontal $\overline{aA}$ and the line Aa_9 drawn tangent to the center line at a_9 represent the directions of the resultant pressures at a and a_9, assuming the pressure at a_9 to be tangent to the center line. From the point A draw $\overline{A2}, \overline{A3}, \ldots$ parallel to $\overline{c_1 c_2}$, $\overline{c_2 c_3}$, ... ; the points of the type c lying in the center of the arch ring; also draw the vertical $\overline{1, 10}$, at such a distance $\overline{A1}$, that $\overline{12}$ is equal to the distance from the soffit at c_1 to the roadway. Then $\overline{23}$, $\overline{34}$, ... are the heights from the soffit to the roadway at c_2, c_3, ... , which may now be laid off as in the figure.

The accurate construction is as follows: the horizontal thrust acting at a must be combined, at the intersection of $\overline{aA}$ with the vertical through c_1, with such a force $\overline{12}$ (force diagram) that the resultant will pass through the center line at a_1. This resultant is, in turn, produced to intersection with the vertical through c_2, where it must be combined with such a force $\overline{23}$ that the resultant $\overline{A3}$ will pass

through a_2, and so on; the forces 1..... 10 being thus found and laid off as before. Now, since the resultants at a_1, a_2, ... are tangent, or very nearly tangent for a segmental arch, to the center line, their directions are evidently parallel to the chords $\overline{c_1\ c_2}$, $\overline{c_2\ c_3}$, . . . as assumed in the first construction.

It is seen from Fig. 8, that a lightening of the spandrel walls, about from a_3 to a_8, conduces to stability. This is often done in large bridges. By this means the ignorant mason who built the Pont-y-Tu-Prydd arch of 140 feet span, 35 feet rise and only 1 foot 6 inches depth of rubble arch ring in the body of the arch, managed to cause the bridge to stand; which when first built fell, by the weight of the haunches forcing up the crown.

On being rebuilt, the spandrels were lightened by cylindrical openings, the spaces between, being filled with charcoal (see VAN NOSTRAND'S MAGAZINE for March, 1873, p. 193) and the bridge stands to this day. Though it evidently does not admit of heavy rolling loads,

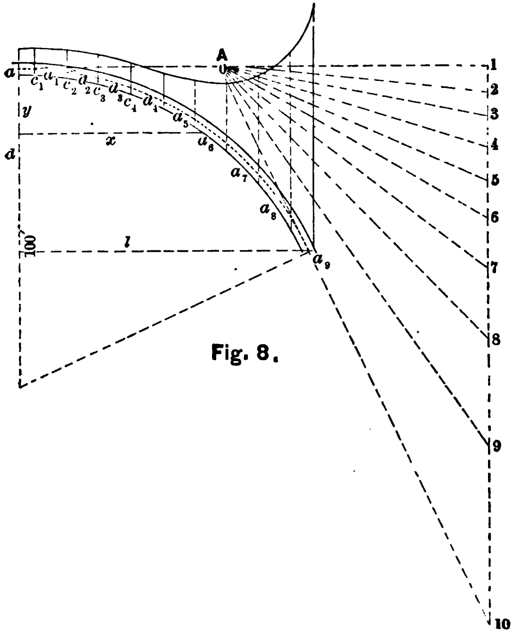


Fig. 8.

it may suffice for a light traffic. As a precedent, however, in construction it is to be avoided.

52. On testing a parabolic arch of 200′ span and 100′ rise in the same way, the line limiting the roadway will be found

to be everywhere the same vertical distance from the soffit; so that where the surcharge is very high the parabola is the best form of arch ring.

This may be shown analytically, in an easy manner. Thus conceive $a_1 a_2 \ldots a_9$ to consist of a thin metallic ring, that is to sustain a uniform horizontal load, w per foot, without bending; required the form of the curve $a \ldots a_9$. It is necessary that the line of pressures coincides throughout with the rib, for if it departs from it at any point, the resultant on that point multiplied by its lever arm to that point, gives a bending moment, which the thin rib is supposed incapable of resisting. (See art. 4 Fig. 2.)

Let $\overline{aA}$ be the axis of X, the vertical down from a the axis of Y. The resultant at a is the horizontal thrust Q. Now take moments of this force, and the downward acting weight on the part aa_6, about a_6, whose coördinates are y and x,

$$\therefore\ M = Qy - \frac{Wx^2}{2}$$

Now M must equal zero for every point of the arch, in which case *the line of pressures will coincide with the figure of the rib.*

Placing $M = o$, we deduce,

$$x^2 = \frac{2Q}{w} y$$

the equation of a parabola, Q.E.D.

53. We see from the foregoing that for a simple arch ring, or for a uniform horizontal load on the ring, and approximately for a very deep surcharge, level at top, the parabola is the best form for the arch ring.

For bridges level at top, at least for bridges whose rise is not over $\frac{1}{8}$ span, the circular is a better form than the parabolic. It is needless to speak of the superiority of the segmental arch over the elliptical and allied forms, whose inherent weakness at the haunches is generally remedied by a greater depth of arch ring, a sufficient reason for choosing

"The rainbow's lovely form"

as the best figure for an arch, however beautiful the elliptical or oval curve may be considered in itself.

54. Fig. 8 will suffice to illustrate the common method, as given by many authors, of finding the curve of pressures. Extend $\overline{aA}$ to c_1, then draw $c_1c_2 \parallel$ A2, which is thus the resultant on joint a_1. Extend this resultant to intersection with vertical through c_2, from which

point draw $c_2c_3 \parallel A3$ and so on. This method, so simple in theory, does not work well in practice, owing to the fact that any error made in finding any point of the curve of pressures is carried on; whereas by the method given in art. 2, any error made is confined to the joint where it is made. It is evident that by the latter method, using *straight* edged rulers and triangles (steel are the best), feathered edged scales, prickers, hard pencils and smooth paper, that the centers of pressure should be found almost to a very needle point. Such accuracy is moreover essential to properly testing an arch ring. It is well to observe that the constructions can be made by drawing only a very few lines. In fact too many lines only serve to confuse the drawing and should be avoided.

CURVE OF PRESSURES CORRESPONDING TO THE MINIMUM AND ALSO TO THE MAXIMUM HORIZONTAL THRUST IN AN UNSYMMETRICAL ARCH.

55. These cases were not demonstrated for an unsymmetrical arch in Part I,

though they offer no difficulty, and are essential to a complete investigation of the stability of an arch. In Fig. 9, *eca* represents an unsymmetrical arch, or an arch acted on by forces, not symmetrical —vertical or inclined.

Let P = resultant of the external forces, acting on the arch between *a* and *c*, not including the reaction R at *a*. Then on combining $R = \overline{ak}$, with P, we get the center of pressure *c* on the joint cc_1. Similarly we could proceed for other points *b*, *d*, *e*, of the curve of pressures,

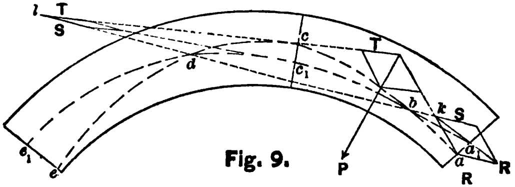


Fig. 9.

corresponding to the resultant $R = \overline{ak}$, acting through *a* in the direction $\overline{ak}$.

Let $a_1\ b\ c_1\ d\ e_1$ be a second curve, corresponding to the reaction R′ at a_1. Now *if* S *is such a force, acting towards the left, that when combined with* R, *it gives*

R′ *as a resultant*, we can find a point c_1, on joint cc_1, of the new curve of pressures, either by combining R′ with P as before, or by combining its components with P: thus call the resultant of R and P, T; this combined at l with S, gives a resultant which cuts joint cc_1 at c_1, *a point lying between* $\overline{kl}$ and c, $\overline{kl}$ being in the direction of s produced.

56. By this construction, it is seen that the new curve of pressures, corresponding to the reaction R′ at a', *passes through b and d* the points where kl intersects the first curve of pressures; for other joints, as ee_1, the new curve lies nearer $\overline{kl}$ than the first curve; since when S acts to the left, the combination of T, for any joint, with S, gives a resultant acting *between* T and S, which therefore cuts the joint nearer $\overline{kl}$ than the first center of pressure.

The above supposes that neither R nor S are vertical, but that both act to the left, whence the horizontal component of R′ exceeds that of R. The joints are, moreover, not supposed inclined more

than 90° from the vertical counting from the top.

57. *Prop. If two curves of pressure cut each other, the curve which lies nearest the straight line, which joins their common points, corresponds to the greatest horizontal thrust.*

We have seen in the preceding article that the two curves *can only* intersect on the straight line *kl* (Fig. 10) as implied in the proposition.

Now if, at any joint cc_1, the center of pressure c_1, corresponding to the curve $a_1bc_1de_1$, lies nearer $\overline{kl}$, the straight line joining *b* and *d*, than the curve *abcde*, then we may suppose a force S, acting in the direction $\overline{kl}$, to be combined with T at *l*, to effect it. The force S, thus found, must therefore when combined with R at *a* give R′; since R and S produce the some effect as R′; so that all points of the first curve can be found by combining R with the resultant of the forces P, up to the joint, and afterwards combining their resultant with S.

The force S, acting to the left, increases

the horizontal component of the resultants on each joint; hence the curve $a_1bc_1de_1$ corresponds to greater horizontal thrusts than the curve *abcde*, as stated in the proposition.

If the arch is symmetrical, the curves of pressure are symmetrical with respect to the crown, whence $\overline{kl}$ must be horizontal, whence follows the conclusions of art. 4, Part I, demonstrated there in another manner.

58. 1/. *If a curve of pressures has two points common to the intrados and an intermediate point common to the extrados, it corresponds to the minimum horizontal thrust.*

For, suppose the curve *abcde*, Fig. 9, touches the extrados near *c*, the intrados on both sides nearer the abutments.

Then any other curve of pressures, a_1 bc_1 de_1 that remain in the arch ring, must cut the first, only in points on the straight line $\overline{kl}$, joining any two points of intersection.

Now the new curve, near the points of contact of the first curve with the con-

ed. The second case of the minimum is but rarely, if ever, found in practice; and of course does not occur when the first case obtains.

61. For the untrue hypothesis of incompressible voussoirs, that curve of pressures which corresponds to the minimum of the thrust is, by the principle of the least resistance, the true one.

We have previously explained in art. 27 how this principle is modified for materials used in practice, both by their compressibility and by the cutting of the stones.

We infer, from the reasoning in art. 27, that with well fitting stones and unyielding abutments, that the actual line of pressures in an arch is found within limits, approximately equidistant from the center line of the arch ring, and having the characteristics of both the max. and the min. of the thrust within those limits. Thus in Fig. 12, the dotted line represents the actual line of pressures within the limits drawn, since the

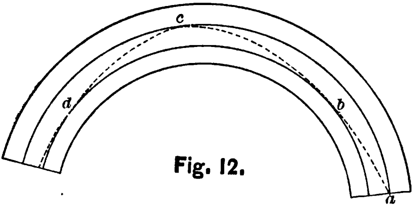


Fig. 12.

part *abc* corresponds to the max. and *bcd* to the min. of the thrust within those limits; for *b* lies above a straight line drawn from *a* to *c* (art. 58, 3/), and *c* lies between *b* and *d* (art. 58, 1/). If the abutments yielded however, the true curve is found within wider limits, and simply corresponds to the min. of the thrust within those limits (see art. 28).

In view however of the influences of temperature, not fitting the stones perfectly, yielding of piers and abutments and shocks, it is recommended to require *in designing an arch*, that for any position of the rolling load, the max. and min. line of pressures may be drawn within somewhat narrower limits than the middle third, otherwise increase the depth an amount, to be left to the judgment of the engineer, as suggested in art. 29.

PRESSURES PER SQUARE UNIT IN EXISTING BRIDGES.

62. By computation, it is found, that in existing bridges, the pressure per square foot on the voussoirs at the crown, *supposing this pressure uniformly distributed*, varies from less than one ton per square foot, for the smallest arches, to 20 tons per square foot and over, for arches of 150 to 200 feet span; the strains being estimated for dead load only.

The normal pressure per square foot at the abutment is greater, often three or four times the above.

It is thus the practice to increase the unit strains with the span.

The material should not be strained at the most compressed edge more than one fifth the crushing weight. Therefore, at the joints of rupture, where the resultant is supposed to pass one-third depth joint from an edge, and the pressure per square foot at the most compressed edge is *double the mean*, the material should not be strained to more than one-tenth

the crushing weight, *if* the pressure is supposed uniformly distributed. On this supposition, if we take the crushing weights of sandstone, limestone and granite (which vary between wide limits) as 300, 400 and 500 tons per square foot, respectively, the allowable strains at the most compressed edges will be 30, 40 and 50 tons. If weaker materials are used the unit strains must be less, and we should increase the depth of voussoirs.

If the curved courses of bricks are in concentric rolls, without bond, the determination of the line of pressures, as well as the distribution of the pressure on each course, becomes uncertain and indeterminate. As a rough guess, if there are n rolls, each roll may be supposed to bear $\frac{1}{n}$ of the pressure. As the outer roll has the greatest span, it is only necessary to test its stability under $\frac{1}{n}$ the total load. It is not recommended, though, to trust to any such rule, but to

bond the rolls, using strong cement; so as to approximate the structure to a "solid arch."

ARCHES WITH VERTICAL AND HORIZONTAL LOADS.

63. In the arch ADB, Fig. 13, suppose it required to pass a curve of pressures through the points A, E and B.

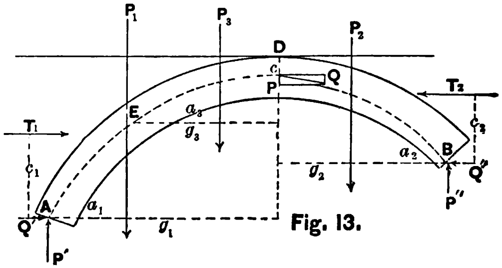


Fig. 13.

Let C be another point of this curve, at the crown, where the horizontal component of the pressure is Q, the vertical component P. Call the vertical components of the loads on the segments AD, DB and ED, P_1, P_2, P_3, respectively; their *horizontal* components T_1, T_2, T_3, respectively.

Call the perpendicular distances from P_1 and T_1 to A, a_1 and c_1; from P_2 and T_2 to B, a_2 and c_2; and from P_3 and T_3 to E, a_3 and c_3, respectively.

Also call the vertical distances of C, the point of application of the inclined thrust at the crown, above A, B and E, b_1, b_2 and b_3, respectively; and the horizontal distances of the same points A, B, E, from the crown, g_1, g_2 and g_3.

64. We now take moments in turn about A, B and E. In eqs. (9) and (11), we suppose the arch to the right of the crown removed, and its effect replaced by the resultant of P and Q acting to the left, P being + when acting upwards; in eq. (10), the part left of the crown is supposed removed and a force equal and directly opposed to the resultant of P and Q acting to the right.

We thus find:

$$a_1P_1 - g_1P + c_1T_1 = b_1Q \quad . \; . \; . \; . \quad (9)$$

$$a_2P_2 + g_2P + c_2T_2 = b_2Q \quad . \; . \; . \; . \quad (10)$$

$$a_3P_3 - g_3P + c_3T_3 = b_3Q \quad . \; . \; . \; . \quad (11)$$

Equating the values of Q in (9) and (10), we find,

$$P=\frac{b_2(a_1P_1+c_1T_1)-b_1(a_2P_2+c_2T_2)}{b_2g_1+b_1g_2} \quad . . (12)$$

From (9) we obtain,

$$Q=\frac{a_1P_1-g_1P+c_1T_1}{b_1} \quad . . . . (13)$$

These equations suffice to determine P and Q, when the position of C is known. When, however, we can only locate the points A, E and B, the values of P and Q and the position of C is found as follows.

For convenience let us make the following abbreviations:

$$g_2+g_3=d_1,\quad g_1-g_3=d_2,\quad g_1+g_2=d_3,$$
$$b_2-b_3=e_1,\ b_1-b_3=e_2,\ b_1-b_2=e_3.$$

Now subtract (10) from (9),

$$a_1P_1-a_2P_2-Pd_3+c_1T_1-c_2T_2=Qe_3 \,. \quad (13a)$$

also, subtract (11) from (9)

$$a_1P_1-a_3P_3-Pd_2+c_1T_1-c_3T_3=Qe_2.$$

Equating the values of Q drawn from these last two equations, and noting that

$a_1P_1(e_2-e_3)=a_1P_1e_1$; $c_1T_1\,(e_2-e_3) = c_1T_1e_1$

we have,

$$P=\frac{e_1(a_1P_1+c_1T_1)-e_2(a_2P_2+c_2T_2)+e_3(a_3P_3+c_3T_3)}{e_2d_3-e_3d_2} \quad \ldots (14)$$

Substituting in eq. (13a) the value of P just found, reducing the terms of one member to the same denominator, collecting like terms, whose coefficients are of the type ed, and noting that, $e_2-e_1=e_3$ and $d_3-d_2=d_1$, we have,

$$Q=\frac{d_1(a_1P_1+c_1T_1)+d_2(a_2P_2+c_2T_2)-d_3(a_3P_3+c_3T_3)}{e_2d_3-e_3d_2} \quad \ldots (15)$$

From (9), we have,

$$b_1=\frac{a_1P_1-g_1P+c_1T_1}{Q} \quad \ldots (16)$$

to fix the position of C at the crown.

We have always for the reactions at A and B, $P'=P_1-P$, $P''=P_2+P$, $Q'=Q-T_1$, $Q''=Q-T_2$.

65. The above equations apply directly to unsymmetrical arches, solicited only by vertical forces by making T_1, T_2 and T_3 zero. Compare Part I, art. 12.

When the arch and load is symmetrical, P=O. If the point of application at the crown is known, we have from (13),

$$Q=\frac{a_1P_1+c_1T_1}{b_1} \quad . \ . \ . \ (17)$$

If two points A and E are given, we have then $g_1=g_2$, $h_1=h_2$, $d_3=2g_1$, $e_3=o$ $P_1=P_2$, $T_1=T_2$, $a_1=a_2$, $c_1=c_2$; whence from (15),

$$Q=\frac{a_1P_1+c_1T_1-(a_3P_3+c_3T_3)}{e_3} \ . \ . \ . \ (18)$$

The position of Q is then found from (16) by making $P=o$.

Eq. (18) is very easily deduced independently.

66. *Application to Underground Arches.*

Let Fig. 14 represent a culvert, with the embankment above it partially completed; so that when the material of the embankment is reduced to the same specific gravity as that of the arch, a line $\overline{ai}$ will limit its top; the earth being level to the left of a and to the right of i.

If the surcharge is of the same specific

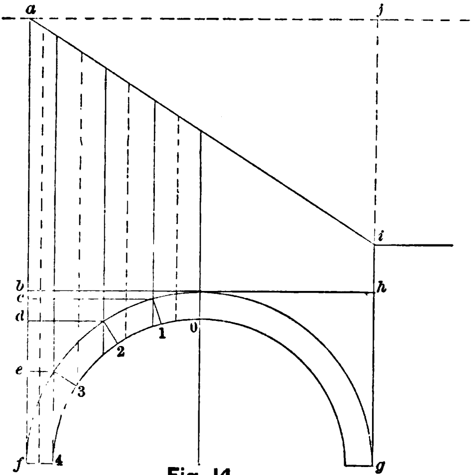


Fig. 14.

gravity up to *bh*, as the voussoirs, then if the earth has a natural slope, the line $\overline{ai}$ will be straight, as drawn; otherwise it may be curved.

The tables for the *vertical forces* are made out as usual. The mean heights of the trapezoids are represented by the dotted lines and the sum of the first three trapezoids will be considered as the surface from the crown to the third joint; similarly for other joints.

This is a sufficiently near approximation for a deep surcharge. For greater accuracy the method detailed in art. 31 may be used.

The horizontal forces are due to the earth pressure and are very difficult to estimate exactly. In a mass of earth with an *unlimited level surface*, the horizontal pressure per square unit at a depth x*

$$p = wx\frac{1-\sin.\ \Phi}{1+\sin.\ \Phi} = wx\ \tan.^2\ (45^\circ - \tfrac{1}{2}\Phi).$$

When the upper surface is at the angle of repose Φ, the pressure per squre unit, parallel to the slope, is,

$$p' = wx \cos.\ \Phi.$$

w represents the weight per cubic unit of the earth.

These formulæ are modified, when the earth is not of unlimited extent, the friction of the abutting surfaces causing a change in the direction of the pressure.

Again, the surface above is sloping from a to i, and level elsewhere.

---

* See Rankine's Civil Engineering, p. 322.

Cohesion, likewise plays an important *role* in earth pressure; its influence becoming much more marked as the embankment grows older. For new embankments it is well to neglect it.

Let us assume, as an approximation, that the horizontal pressures, due to the earth, on voussoirs 1, 2, 3 and 4, are due to the heights x measured along the dotted lines from the extrados of each voussoir to the surface of level topped earth.

The surfaces against which these pressures act for voussoirs 1, 2, 3, 4, are, $\overline{bc}$, $\overline{cd}$, $\overline{de}$, $\overline{ef}$, respectively; so that the horizontal pressure acting on the third voussoir, for instance, is equal to the product of the height $\overline{de}$, by the height of the surcharge from the extrados to the surface, by $tan^2\ (45-\frac{1}{2}\Phi)$, ($w$ being taken as unity). In the following examples let $\Phi=30°$, so that, $tan^2\ (45-\frac{1}{2}\Phi)=\frac{1}{3}$.

The horizontal pressure then upon the third voussoir is, $\overline{de}\times x\times\frac{1}{3}$. It may be written $\frac{yx}{3}$ for any voussoir. The lever

arms of these forces, about the top of the arch, are the vertical distances from the line $\overline{bh}$ to the middle of the segments $\overline{bc}$, $\overline{cd}$, $\overline{de}$ and $\overline{ef}$. The moment of these forces, down to any joint, divided by the sum of the same forces, gives the vertical distance from the line $\overline{bh}$ to the resultant of the forces taken, as given in the last column of the following tables concerning horizontal forces.

67. *Example.*—Let the span of the semi-circular culvert be, 11.30 units of length, the depth of voussoir 0.94, the height $\overline{ab}$ of the reduced surcharge 25.12, and the height $\overline{hi}$, 12.56. The filling up to $\overline{bh}$ is taken of the same density as the voussoirs. If the backing was solid up to $\overline{bh}$, the horizontal forces would be due more nearly to the depth from a to the voussoirs on the left, and from i on the right.

Each of the semi-arcs with its load is divided, as shown in Fig. 15, into eight parts (approximate trapezoids), of which the first six have a width of 0.94, the two last a width of 0.47.

In the following table for vertical forces, column (1) gives the joint, columns (2) and (3) the force from the crown to the joint and its lever arm, respectively, for the left semi-arch, columns (4) and (5), giving the same quantities for the right semi-arch:

	Left Side.		Right Side.	
Joint.	Force.	Lever Arm.	Force.	Lever Arm.
(1)	(2)	(3)	(4)	(5)
1	18.98	0.47	18.25	0.47
2	38.99	0.95	34.07	0.93
3	60.24	1.45	50.98	1.39
4	82.79	1.95	67.57	1.85
5	106.87	2.47	84.16	2.31
6	133.23	3.01	101.28	2.79
7	147.84	3.29	110.58	3.05
8	162.63	3.57	119.68	3.29

The next table refers to the horizontal forces; column (1) referring to the joint, column (2) gives the product $\frac{1}{3}yx$ (see art. 66), column (3), its lever arm about the summit, column (4), the moment, col-

umns (5), (6) and (7) the sum of the forces down to any joint, their moment and the distance of their resultant below the line $\overline{bh}$, respectively.

	Left Side.			Horizontal Forces.		
Joint.	Force.	Lever Arm.	Moment.	Force.	Moment.	Lever arm.
(1)	(2)	(3)	(4)	(5)	(6)	(7)
1	0.58	0.05	0.03	0.58	0.03	0.05
2	1.29	0.19	0.25	1.87	0.28	0.14
3	2.73	0.47	1.28	4.60	1.56	0.34
4	4.35	0.94	4.09	8.95	5.65	0.63
5	6.37	1.62	10.33	15.32	15.98	1.04
6	11.09	2.64	29.28	26.41	45.26	1.71
7	9.40	3.77	35.44	35.81	80.70	2.25
8	23.46	5.43	127.39	59.27	208.09	3.51
	59.27		208.09			

Columns (1), (5) and (7) are next given for the right side.

Right Side.			Horizontal Forces.		
Joint	Force.	Lever Arm.	Joint	Force.	Lever Arm.
(1)	(5)	(7)	(1)	(5)	(7)
1	0.59	0.05	5	11.56	0.99
2	1.64	0.14	6	18.51	1.62
3	3.80	0.33	7	24.17	2.12
4	6.96	0.60	8	38.25	3.34

Now let it be required to pass a curve of pressures, 0.41, below the top of the crown joint, and through the lower middle third limits, at joints six on either side.

Now the vertical loads from the crown to joints six on left and right are (see table) $P_1=133.23$, $P_2=101.28$; the distances of their resultants from the vertical through the crown are 3.01 and 2.79 respectively; whence by measurement on the drawing, $g_1=g_2=5.1$ and $a_1=5.1-3=2.1$, $a_2=5.1-2.8=2.3$.

Similarly,

$$T_1=26.41,\ c_1=3.5-1.7=1.8$$
$$T_2=18.51,\ c_2=3.5-1.6=1.9$$

whence by eq. 12, art. 64,

$$P=\frac{a_1P_1+c_1T_1-(a_2P_2+c_2T_2)}{2g_1}=5.8$$

Also by eq. (13), $Q=96$.

Now lay off on vertical lines, $\overline{08}$, to left and right of the center, the numbers in columns 2 and 4 respectively, being the vertical loads from the crown to the joints in order. From columns (3) and (5) of the same table lay off the distances, on the horizontal through the summit, from the crown to the centers of gravity of the vertical loads in order. Thus $\overline{S6}=3.01$ corresponding to $P_1=133.23$.

Next, from the tables referring to horizontal forces, lay off on the horizontals through 1, 2, . . . , to 1′, 2′, . . . , the forces given in columns 5, for the left and right side respectively. Also lay off on vertical lines the numbers in columns (7), measuring from the line $\overline{gS}$. Thus the total horizontal earth thrust from the crown to joint 6 on the left is $T_1=26.41$; and its point of application is $\overline{g6}=1.71$ below the summit. To

find the thrust at the crown, lay off $\overline{mn}=Q$ horizontally, and $\overline{no}=P$ vertically downwards: $\overline{mo}$ ($= vo$ drawn $\parallel mo$) is then the resultant at the crown joint in position and magnitude. Draw the lines $v_1', v_2', \ldots$ Now, to find the center of pressure on a joint, as the 6th on the left, draw vertical and horizontal lines $\overline{6b}$, $\overline{6b}$, through the points of application of P_1 and T_1, to intersection b; which is thus the point of application of the resultant of P_1 and T_1, represented by a straight line from 0 to $6'$ in the force polygon on the left. From b draw $ba \parallel \overline{06'}$ to intersection a with $\overline{mo}$ produced; from a draw $\overline{ac} \parallel \overline{v6'}$ to intersection with joint 6 at its center of pressure. It is evident that the resultant there is represented by the line $\overline{v6'}$, the resultant of $\overline{06}$, $6\bar{6}'$ and $\overline{vo}$, or of P_1, T_1 and the inclined thrust at the crown; similarly on the right side, to find the position of the resultant on joint 8, we find d, 3.29 to the right of S and 3.34 below it; thence draw $de \parallel 0\bar{8}'$ to intersection e with $\overline{mo}$ produced; thence draw $\overline{ef} \parallel \overline{v8'}$ to f the required

point; the magnitude and direction of the resultant being represented there by the line $\overline{v8'}$.

The line of pressures thus found, represented by the dotted line, leaves the middle third at joints 4, 5 and 8 on the right, and at joint 8 on the left.

By raising the point m nearly to the upper middle third limit, and the center of pressure at joint 5 on the right to the lower middle third limit, the curve of pressures will remain within the inner third except near the abutment. The arch stones should be increased in depth there, up to about joints 6.

The earth is next supposed *level at top*, the distance $ba=hj$, Fig. 14, being 25.12 units. On making out a table of vertical and horizontal forces as before, for one side only, we find from eq. (17), art. 65, $Q=122.2$ cubic units of stone, on passing a curve of pressures through the upper middle third limit at the crown and the lower middle third limit at joint 6.

The curve thus found keeps every-

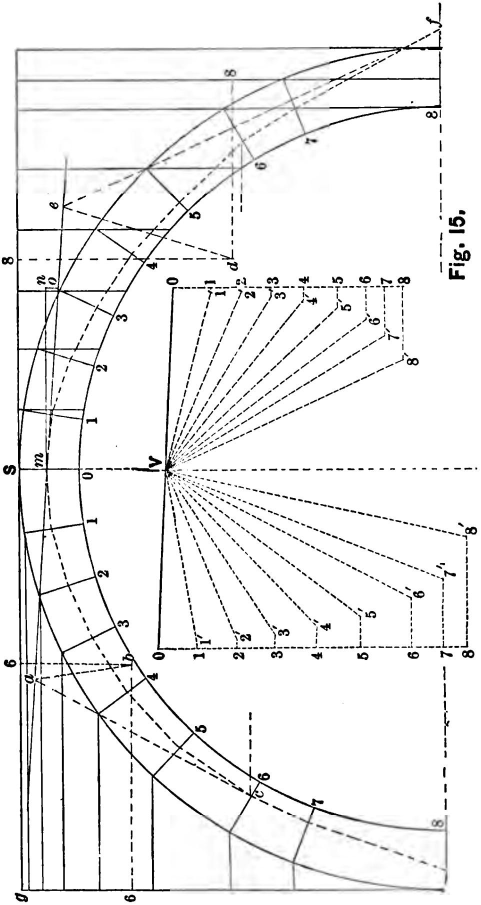


Fig. 15.

where in the middle third except at joints 8, where it nearly reaches the extrados.

68. If the arch stones are not increased in depth near the abutment, joint 8 will tend to open at the intrados; but this it cannot do unless the haunches spread; which is in turn resisted by the spandrels; or if there are none, by an increased horizontal thrust which the earth is capable of putting forth, thus keeping the line of pressures within the arch ring, *e.g.*, within the middle third if the deformation that the earth permits is small.

Experience shows that very thin arch rings, built in rubble, often can fulfill the conditions of stability when embanked over carefully; the centers being struck after the embankment is mostly completed.

In such cases the earth must exert larger horizontal forces, than given above; so that it is well to be guided mainly by experience in designing underground arches as before remarked.

By increasing the depth of arch stones

near the abutment, as suggested, we are safe in presuming on stability without the aid of extra horizontal forces over the ordinary active earth thrusts (see art. 50).

69. The dimensions of the preceding culvert and surcharge may be taken in any unit as feet, meters, etc.

If the unit taken is the meter it corresponds to a railroad culvert at Schwelm, the top of the embankment being 31.^{m}40 above the top of the arch, corresponding to a weight 25.^{m}12 high of materials as dense as the voussoirs, as given by Scheffler. From the diagram for the earth level at top, we find that the normal components of the pressure on joints 6 and 8 are about 180 cubic meters of stone; so that, if uniformly distributed, the pressure per square meter would be $\frac{180}{0.94}=192$ cubic meters of stone.

If we take the weight of a cubic foot of stone at 140 pounds (very low), since there is about 35 cubic feet in one cubic meter, 192 cubic meters $=140\times35\times192$

pounds=448 tons. This pressure being on one square meter=10.8 square feet, the pressure per square foot=41.5 tons. If the line of pressures, at joints 6 and 8, is $\frac{1}{8}$, $\frac{1}{4}$ depth joint from edge, the pressure per square foot at the most compressed edges is 83, 111 tons respectively (see arts. 21, 22).

If the material is nowhere to be subjected to more than $\frac{1}{8}$ the crushing weight, it is evident that the best granite or limestone should have been employed in this bridge, having a crushing weight of 400 to 500 tons per square foot. It is stated that the material was not of a good character, and that a large number of voussoirs in all parts of the arch were crushed in consequence.

If the unit of length taken is the foot in the preceding example, we see, that a semi-circular arch of 11.3 feet span, and 1 foot depth of voussoir, is stable against rotation, when the top of the embankment is about 30 to 40 feet above the crown, provided the voussoirs are increased in depth towards the springing,

say to 2 feet. The normal pressure now on joint 6 is 180 cubic feet stone=11.3 tons per square foot. At the most compressed edge the pressure is 22.6, if no joint opens, which even sandstone can very well stand.

Any multiple of these dimensions, as 22′.6 span, 2′ depth voussoir and 60′ to 70′ height of surcharge, will show equal stability against rotation. The thrust is now 180 cubic units of stone=1440 cubic feet of stone=90.4 tons. This acts on a surface of 4 square feet; so that the uniform compression is 22.6 tons and the compression at the edge of joint 6 is 45.2 tons about; which sandstone can again bear.

70. It is evident that the height of surcharge should be considered in designing culverts; though it is neglected in the practical formulæ proposed by some authors.

Trautwine (Engineers' Pocket Book p. 347) says:

We have known nearly semicircular arches of 30 to 40 feet span, to be thus built successfully (*i.e.*, when the centers are left standing

until the earth-filling is completed above the culvert) with scarcely a particle of masonary above the springs to back them." He recommends *not to do less*, and that only in small spans, than make the height of backing above the springing over the abutment, ½ the total height of the arch from the springing to the top of keystone; and from the point so found to draw a tangent to the arch to limit the backing. As suggested above, it is still better to increase the depth of voussoirs towards the abutment.

Rankine well suggests that "over the arches of culverts, the earth rammed in thin layers should rise to at least half the height of the proposed embankment; the remainder may be tipped in the usual way."

71. It is more than probable that in culverts, where the loose earth is deposited after the culvert is built, especially if the centers are not struck until the embankment is completed over the arch, that the *entire height* of surcharge *does not* press upon the culvert. Cohesion and friction both influence the result. Thus if the grains of earth had the cohesion of stone, the *slight sinking of the culvert*, due to its compressibility, would relieve it of part of the load above

it, especially in small culverts. The same sinking would cause a portion of the weight above the arch to be transmitted by friction to the sides.

Thus let s=span of culvert, h=height of surcharge above its top. Now the horizontal pressure of the earth at a depth x on a surface Δx being, wx tan.^{2} $(45° - \frac{1}{2}\,\Phi)\,\Delta x$ nearly, the limit of the sum of quantities of this type between the limits, $x=h$ and $x=o$, $=\frac{wh^2}{2}$ tan. $(45° - \frac{1}{2}\,\Phi)$, is the total horizontal earth thrust exerted on a vertical plane 1 foot wide and h deep below the surface.

If the culvert were suddenly removed, the mass above, if it had no friction or cohesion, would, like a perfect fluid fall, the top surface changing to a lower level. Now consider friction alone, its coefficient being called f=tan. Φ. The weight of the mass above the arch is, wsh; the friction of the vertical parallel walls on either side is $2f\frac{wh^2}{2}$ tan.$^2(45° - \frac{1}{2}\Phi)$

Now if the arch yields somewhat, the weight still sustained by it is the difference between these expressions.

There is no weight sustained by the arch for,

$$h=\frac{s}{f\,\text{tan.}^2\,(45° - \frac{1}{2}\Phi)}$$

and for one half this height, the difference above gives the maximum weight sustained by the lowered arch.

Thus if $s=15$ feet and $\Phi=34$, there is no weight on the arch for $h=75$ feet; the maximum load obtains when $h=37\frac{1}{2}$ feet. In the first case, the weight above the arch is entirely spread to one side.

In reality the lower particles descend, as in a beam, so that a wedge-shaped mass would probably fall down, the material above forming a natural arch. This often happens in brick walls with arched openings in them; the arch yielding so that a natural arch is formed above it, and as a consequence the arch does not sustain the whole weight of surcharge. This happens over every lintel or other compressible support at the top of windows doors etc. in the walls of houses, churches, etc.

The analysis above is given only to illustrate partially the principle enunciated, and not for use in practice.

72. The weight resting on a culvert is not that due to the total height of surcharge for another reason. Draw the trapezoidal cross section (perpendicular to the roadway) of the embankment, and divide it by a vertical line into two equal parts. In consequence of the symmetry

the earth thrust of one-half of the slice shown by the cross section (supposed to have any width) against the other half, is horizontal. On combining this thrust with the weight of one-half of the slice acting at its center of gravity, the resultant of course strikes the base farther from the center than if there were no horizontal thrust. Its effect is evidently to increase the vertical pressure towards the slopes and diminish it near the center of the cross section, Q.E.D.

It is not considered advisable to proportion large culverts for a less weight than that due to the whole surcharge; but small drains may be made smaller than such theory requires.

73. *The abutments* of culverts are treated in the graphical construction exactly as though they were a part of the arch. The horizontal thrust of the earth on their outside is not always that due to the height of surcharge, as the ground generally rises abruptly from the foundation of the abutment walls.

It is best, then, to estimate for both

cases: the earth pressing and not pressing against the back of the culvert. Again, as part of the weight above the arch is transmitted to the sides, the weight sustained by the abutments may be increased; also the horizontal pressures acting on them and the arch. It is best to err on the same side in designing these structures.

74. *Tunnel arches.*—In treating tunnel arches, only a part of the weight above them is supposed to press on the arch, the balance being transmitted to the sides. Cohesion now plays the most important part. Tunnels have been executed, even in clay, that have stood for some time without support. In such tunnels large masses often fall in, completely choking up the tunnel; so that if an arch of wood, stone, or other material was built before the fallen mass lost its cohesion it would eventually have to support all, or a part of its weight. What weight presses on a tunnel arch cannot be estimated; we can only resort to experience here.

Rankine gives the following formula,

founded on practice, for the minimum thickness, t of tunnel arches,

$$f=\sqrt{.12\,r},\ r=\frac{a^2}{b^2};$$

where a=rise and b=half span.

"This is applicable where the ground is of the firmest and safest kind. In soft and slippery materials, the thickness ranges from $\sqrt{.27r}$ to $\sqrt{.48r}$."

75. The *arch* is peculiarly adapted for a tunnel support; for it is the great advantage of the arch, that it *will not be forced in* at one place *without it is forced out* at another. The latter, the enveloping mass generally prevents, if stones and earth are packed in tight back of the arch; so that the arch so constructed should generally stand unless crushed from a too heavy load.

As in practice, tunnel arches are not thus crushed, we may infer, as stated before on theoretical grounds, that only a part of the superincumbent material presses on them. In every deep tunnel, the thickness of the arch ring is not increased over that due to a compara-

tively small height, as is inferred from the preceding formula.

If a quicksand is encountered on one side or the other, the curvature of the arch must be sharply increased there, or the arch may be forced in, as has happened in certain treacherous clays.

76. Assuming the preceding formulæ for earth thrust and the depth of surcharge that is supposed to press, in accordance with this theory, the stability of a tunnel arch is investigated as previously explained in the case of culverts.

Thus take the tunnel arch under the Thames, Fig. 16; whose dimensions in meters are as follows: the thickness of the arch ring is about 0.94, the radius of the upper part $\overline{0..6}$ is 2.16, and of the inferior part $\overline{6..10}$, 8.61 meters; the upper part being formed of three concentric rolls without bond.

The earth and water above the tunnel is supposed to exercise upon the arch a pressure corresponding to a load $7^{m}.54$ high of material like that of the voussoirs; the reduced surcharge being supposed level at top for simplicity.

The upper part of the arch is divided into six parts, having widths of 0.90, 0.63, 0.63, 0.31, 0.31, 0.31, respectively. The lower part is divided into four parts, having lengths, along the center line, 0.47, 0.76, 0.75 and 0.94 respectively and whose lever arms are the distances of their centers of gravity from the vertical axis of the tunnel, or from the horizontal through the top of the arch, for the vertical or horizontal forces respectively. The tables are made out as in art. 67. If preferred, the voussoirs and surcharge may be considered separately as in art. 31, to attain greater accuracy; but the usual method elsewhere followed is sufficiently near for the object in view.

The method followed is moreover as correct for voussoirs 7 to 10 as the one followed in art. 31, since the particular division of the arch preceding any voussoir is immaterial as concerning that voussoir.

In the following condensed table the first column gives the joint, the next the vertical force on that joint counting from

the crown, the third column its lever arm counting from the vertical through the crown; columns 4 and 5 give the horizontal force (acting on the extrados of the arch from the crown to the joint considered) and its lever arm about the top of the arch respectively.

Joint.	Vertical Forces.		Horizontal Forces.	
	Force.	Lever arm.	Force.	Lever arm.
1	7.67	0.45	0.38	0.075
2	13.25	0.77	1.11	0.22
3	19.24	1.11	2.45	0.47
4	22.54	1.28	3.54	0.66
5	25.84	1.46	5.40	0.99
6	29.14	1.60	9.38	1.63
7	29.58	1.64	11.08	1.89
8	30.29	1.66	14.23	2.35
9	30.99	1.68	17.39	2.82
10	31.87	1.70	22.49	3.49

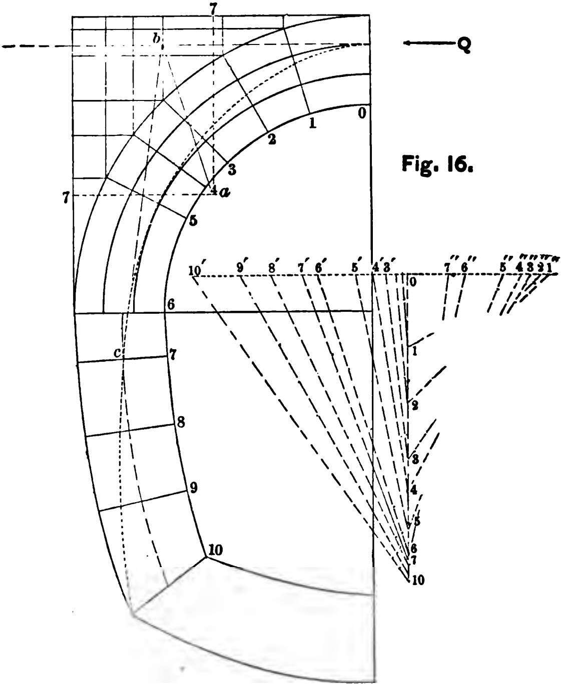
To pass a curve of pressures through the upper middle third limit at the crown and the lower middle third limit at joint 5, we have by measurement and from the tables,

$$Q = \frac{aP_5 + cT_5}{b} = \frac{.79 \times 25.84 + 1.02 \times 5.40}{1.69} = 15.34.$$

Now lay off the line of vertical loads, $\overline{0\ .\ .\ 10}$, from column 2, and the line of horizontal forces, $\overline{0\ .\ .\ 10'}$ from column 4.

Next, on the horizontal, through the summit, lay off the lever arms in column 3; and on the vertical, tangent to the extrados, the lever arms in column 5; also from each point, 3′, 4′, 5′, . . . lay off to the right $\overline{3'\,3''}$, $\overline{4'4''}$, $\overline{5'5''}$, . . . , each equal to Q.

To find the resultant of the presssure on any joint, as the 7th, we draw $\overline{7a}$, $\overline{7a}$, representing the positions of the vertical and horizontal forces ($\overline{07}$, $\overline{07'}$, force diagram) to intersection a; from a draw $\overline{ab} \parallel \overline{77'}$ to intersection b with Q prolonged; at b draw $\overline{bc} \parallel \overline{77''}$ to intersection c with joint 7, which is thus the center of pressure on that jont. The resultant on joint 7 is represented by the line 77″ of the force diagram, being the resultant of $\overline{07}$, $\overline{07'}$ and Q.



The line of pressures thus found is represented by the dotted line, and leaves the inner third of the arch ring at joints

9 and 10. It does not follow that the arch is unstable. The active forces may not be, and probably are not, as estimated. Whether this be so or not, the joint 10 cannot open inside without the arch being forced out at the haunches; but this, if the earth is packed tight around the arch, is resisted by the earth outside, which there exerts a larger horizontal force than estimated (partly passive), and thereby restricts the line of pressures to narrow limits.

It is thus evident that the arch is stable unless the surrounding earth itself gives way, which will generally not happen. The surrounding mass thus plays the part of spandrel, besides exerting an active thrust, being the least thrust it is capable of. It is even more effective than a spandrel, since it can prevent the crown from rising, which, in stone viaducts, is one method of rupture which the spandrel actually aids in producing.

77. The above construction applies when the rolls of the arch are bonded together. As this is not true in the

present case, the problem of determining the true line of pressures becomes impossible of solution.

If we conceive one-third the vertical and horizontal forces, given in the previous table to act on each roll, we shall find that for the outer roll that a curve of pressures cannot be drawn within the middle third, though one can be drawn within the arch ring from the crown to joint 6.

However, the passive earth thrust in good earth will again prevent deformation and thus cause stability as before; but the arch cannot be considered as strong as if it were bonded throughout so as to act as one mass.

78. The reversed arch at bottom is not supposed to exert any horizontal force, so that the sides of the arch simply rest on it, as though it was an abutment. Otherwise it is used to prevent a forcing in of the sides; and the sum of its horizontal component and that at the crown equal the total horizontal thrust of the earth or fluid on one side.

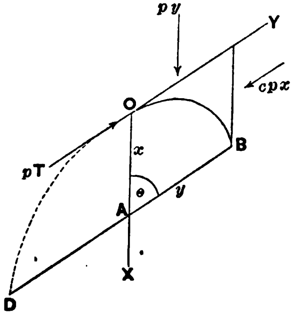
79. The curve of pressures shown in fig. 16 resembles somewhat a *quarter ellipse*; so that if such a curve was taken for the center line of a tunnel arch, a curve of pressures might be drawn very nearly following this center line; so that if the arch was really acted on by the forces supposed, the arch designed would offer a very great stability.

Similarly if the centre of the ellipse is taken at $\frac{1}{8}$ height of arch above the inverted arch, it will be found that a curve of pressures may be drawn very near the center line, especially if the surcharge has a less specific gravity than the arch. In fact, the ellipse is recognized as the proper form for tunnel arches and any part of the curve may be used that will best subserve the purpose in view—the axis of the tunnel always coinciding with that of the ellipse.

From Fig. 16 we see that if a semi-circle is used for the curve of the upper part, that the lower part had best be made vertical on the sides. This form of tunnel arch is quite common in prac-

tice and is commended when good material is used.

REMARK.—We can easily prove analytically that the ellipse is the proper form for a tunnel arch, when the depth of surcharge is so great that the thrust of the earth at any part of the arch is practically the same. To take the most general case, let the top surface of the earth be $\parallel$ OY in the adjoining figure. Call the pressure, per unit of inclined plane OY, in a vertical derectior, p; the conjugate pressure in a direction $\parallel$ OY, per unit of a vertical plane OA, can be represented, according to the theory of earth pressure, by cp, c being a constant. Let $OA=x$, $AB=y$, $OAB=\theta$, and call the thrust at O in the direction OY, tangent to the rib BOD at O, pT.



This rib, or "linear arch," is not supposed to have any bending moments at any point, so

that the thrust at any point is tangent to the rib; otherwise a deformation would ensue, due to the normal component, which is contrary to our supposition of a linear arch. The total vertical pressure on OB is py, the conjugate pressure is cpx. Being uniformly distributed, their lever arms about B are, $\frac{y\sin\theta}{2}$, $\frac{x\sin\theta}{2}$, respectively.

Now if any point, as B, of the arc is to be a point in the line of pressures, we must have, taking moments about B,

$$pTx\sin\theta=\left(\frac{py^2}{2}+\frac{cpx^2}{2}\right)\sin\theta$$

$$\therefore\ y^2=2Tx-cx^2$$

the equation of an ellipse,

Q.E.D.

The equation of an ellipse referred to a diameter and the tangent at its vertex, a and b being the semi conjugate diameters is, $y^2=\frac{b^2}{a^2}(2ax-x^2)$.

Comparing with the above, we have,

$$T=\frac{b^2}{a},c=\frac{cp}{p}=\frac{b^2}{a^2}.$$

Or *the intensities of the conjugate pressures are as the squares of the diameters to which they are parallel.*

If in the eq. above we make, $x=OA=a$, we

find, $y=AB=b$; whence from the last eq., $\frac{a.\ cp}{b.\ p}=\frac{b}{a}$. Now the thrust at $O=pT=acp$, whilst that at the ends of the conjugate diameter DB, acting $\parallel$ OX, is bp; hence *these forces are proportional to the diameters to which they are parallel.*

To construct the arch, c and a or b must be given to find the other semi diameter from the eq., $c=b^2\div a^2$.

WHEN THE TOP SURFACE OF EARTH IS LEVEL, OY becomes level and $\theta=90°$. a and b are now the semi axes of the ellipse. From the theory of earth pressure, $\frac{cp}{p}=\tan.^2(45°-\frac{1}{2}\Phi)$ whence, for a tunnel arch,

$$\frac{\text{horizontal semi axis}}{\text{vertical semi axis}}=\frac{b}{a}=\tan.(45°-\tfrac{1}{2}\Phi);$$

Φ being the angle of repose.

Next, make $c=1$ (θ being 90°) and the ellipse becomes a *circle.* At any point of the circle consider the two equal forces p, p, at right angles and acting on a unit of area of planes $\perp$ to them. Their resultant acts normally to the circle, and its intensity is easily found to be p, the intensity of the vertical and horizontal components.

Calling $r=a=b$, the radius of the circle, we have the thrust at $O=pT=pr$, or *the product of the intensity on a unit of circumference by the*

radius. This thrust is the same all around the ring.

The above are the principal deductions given by Rankine for the arches given above. The reader is referred to his Civil Engineering for the "geostatic" and other forms of linear arches.

GROINED ARCHES.

80. Let ABCD, Fig. 17, be the plan of a groined arch, $\overline{AC}$ and $\overline{BD}$ representing the groins; the elevation is shown, at BMC, of the front face AD. There are abutments at A, B, C and D, one of which is shown at A in plan.

Let us divide the portion of the arch and load between the groins into simple arches, as AID, a_8 IJ, . . . which rest at their extremities on the groins AE, DE. We can estimate the stability of any one of these arches by principles previously established, and find the resultant pressure that it exercises upon the groin. The latter supports a similar pressure from each side; the resultant of these two pressures, which is generally oblique, can then be decomposed into horizontal

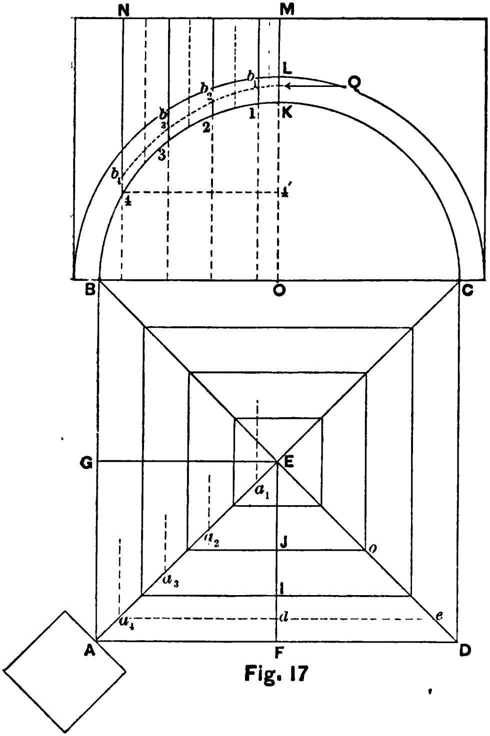


Fig. 17

and vertical components, which are the forces to be used, in their proper posi-

tions, in ascertaining the stability of the simple arch constituting the groin, and also of the abutment against which it leans.

81. An example will render this clearer. The dimensions are given in meters, though any unit may be taken. Let $\overline{AD}=7.54$; the arc AFD in plan, a semicircle whose radius is thus, $\overline{OB}=3.77$; depth of keystone $\overline{KL}=0.47$; the height of surcharge above it, LM=1.26. Divide the semi groin AE into a number of equal parts, four in the figure, and suppose, each simple arch, as AID, to terminate at the middle, a_4, of its corresponding division. Project up a_1, a_2, a_3, a_4 to b_1, b_2, b_3, b_4, in elevation. Then on this supposition the weight AIF sustained at a_4 is represented in elevation by MNb_4K, supposing the joints vertical. Similarly for arch a_3 JI, etc. Pass a curve of pressures now through the top of middle third limit at the crown KL and the lower middle third limit about the joint of rupture, taken approximately at b_4 in this case, from which the resultant at

$(a_4\ b_4)$ can be found. It is sufficiently correct, and is on the side of safety for the other arcs, as a_3 IJ, to retain the same value and position of Q at the crown. We thus find from the diagram for arc AID the resultants in amount, position and direction at the points $(a_3,\ b_3)$, $(a_2,\ b_2)$, $(a_1,\ b_1)$ of the groins, due to all the arcs in the space AEF.

In the following table of volumes and centers of gravity, v = volume of trapezoid lying just to right of joint to which it refers = width × mean height × $\overline{\text{IF}}$.

In this case IF = JI = 0.94. l is the distance of the center of gravity of the trapezoid from the crown, and m the corresponding moment. V is the volume from the crown to the joint to which it refers found by cumulating the numbers in column v. Similarly M is formed from m, and the quotients $\frac{\text{M}}{\text{V}} = \text{C}$, give the distances of the centers of gravity of these volumes V from the crown.

Joint.	v	l	m	V	M	C
1	.78	.24	.1872	.78	.1872	.24
2	1.68	.96	1.6128	2 46	1.8000	.73
3	2.04	1.91	3.8964	4.50	5.6964	1.27
4	2.74	2.87	7.8638	7.24	13.5602	1.87
	7.24		13.5602			

Laying off $1^m.87$ from the crown to the left on Q prolonged and drawing from this point a straight line to b_4, we have the direction of the resultant at b_4. Its amount is found by laying off $P_4=7.24$ on the vertical, through the point 1.87 to left of the crown, downwards, and then drawing a horizontal line to the resultant, which may now, as well as Q, be scaled off. We thus find $Q=5.45$.

Next, combining the forces at a_1, a_2, a_3 and a_4 due to the arcs on either side of the groin AE, we have for the vertical components of the resultants at a_1, a_2, a_3, a_4, 1.56, 4.92, 9., 14.48, respectively, being double the numbers given in column V above.

The horizontal component at each point is $Q\sqrt{2}=7.7$.

82. It is evident that the greater the number of divisions IF, JI, &c., the more accurate the result. It is well to test the above volumes analytically.

Call the radius, $OB=r$, $\overline{KM}=c$, and the variable distance $Ed=a_4d=x$. Since any arc as a_4de of the intrados has a radius r, its *rise*, when the half chord is x, $=(r-\sqrt{r^2-x^2})$.

For simplicity we shall introduce one approximation, viz: that the areas of the circular spandrels is nearly that of parabolic spandrels of same encompassing rectangle. This is on the side of safety.

We have, calling $44'=x$,

$$\text{area KMN } 4=\frac{x(r-\sqrt{r^2-x^2})}{3}+cx.$$

On multiplying this by $\triangle x$ = elementary width of a slice, we get the approximate volume of one half of a simple arch, $\parallel \overline{AD}$, and at a distance x from E. The limit of the sum of such slices, between the limits $x=r$ and $x=o$, gives the total volume of the part AEF

$$\therefore \text{ volume AGFE}=\tfrac{1}{4} \text{ groined arch}$$

$$=\lim \sum_{0}^{r}\left(\tfrac{2}{3}xr-\tfrac{2}{3}\left\{r^2-x^2\right\}^{\frac{1}{2}}x+2cx\right)\triangle x$$

$$=\left(\frac{r}{9}+c\right)r^2$$

In this case $r=3.77$ and $c=.47+1.26=1.73$

$\therefore$ vol. AGFE$=30.53$.

From the table we find this volume to be $2(.78+2.46+4.50+7.24)=29.92$, which differs but slightly from the above, and is somewhat less, as it should be.

83. It will conduce to clearness to lay off on an elevation of the groin and abutment, Fig. 18, the forces just found,

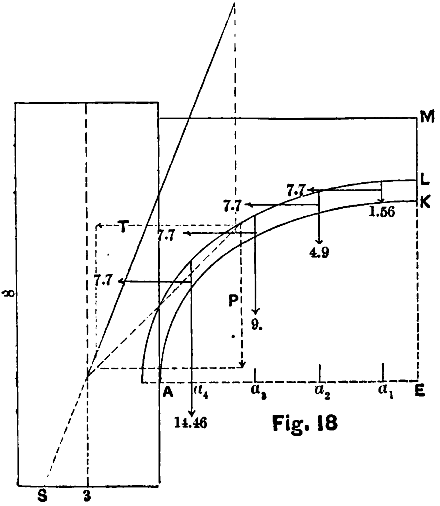


Fig. 18

directly over a_1, a_2, . . . at the same heights as b_1, b_2, . . . are above the springing, viz., 4.03, 3.76, 3.15 and 2.12. The distance t of the resultant T of the horizontal forces above the springing is thus,

$$t=\frac{7.7(4.03+3.76+3.15+2.12)}{T=30.8}=3.26$$

from which T can be located. Similarly take moments of the vertical forces about A, to find the distance, p that their resultant, P, acts to the right of A.

$$\therefore\ p=\frac{14.46\times.68+9\times2.01+4.9\times3.35+1.56\times4.69}{P=29.92}$$
$$=1.73$$

The resultant of T and P passes outside of the arch ring above the springing. On combining it in turn with the weight of the abutment $8\times3\times2$, the final resultant cuts the base at s, $\frac{1}{6}$ the depth from the outer edge. The abutment may be increased in size to cause s to pass as near the center as may be desired.

84. The arch ring of the groin in the actual example has a depth of $0.^{m}94$, being double that of the ring as drawn; which may thus be supposed to represent its middle half.

To test its stability, combine the resultant of the forces 7.7 and 1.56, being the pressure on the joint midway between a_1 and a_2, with the resultant of the next two concurrent forces, 7.7 and 4.9, to find the resultant on the joint midway between a_2 and a_3; next, combine this last resultant with that of the next two concurrent forces and so on. The final resultant on the springing joint should coincide with the resultant of P and T just found.

The line of pressures is thus found to keep very near the center line down to a_4, below which it passes out of the arch ring, on the extrados side.

The heavy backing will exert horizontal forces to modify this line of pressures, probably keeping it in the arch ring near the springing; for otherwise the intrados joints about the springing must open;

but this cannot happen unless the extrados joints open about a_8. If the backing prevents the latter, the former cannot occur; but if no joints open, the line of pressures must lie in the middle third; so that the arch ring is stable.

85. It will be observed, that no pressure at the crown is needed to ensure stability. In fact, if any were supposed it would only cause the final resultant on the springing joint to depart still more from the arch ring.

For other dimensions than those given in this example, a horizontal thrust at the crown of the groin may be needed. For example, when the line of pressures just found falls below the middle third at any joint.

In this case, if we desire that the curve of pressures pass through a particular point in the lower middle third limit, as the one vertically over a_4; let the horizontal thrust, H at the crown KL act at the upper middle third limit. Call the lever arms of H, T and P about the lower point vertically over a_4, h, t and p; we have,

$$Hh+Tt-Pp=o$$

from which H can be found and the curve of pressures located as before.

The true curve keeps within the middle third, and, as before explained, conforms nearly to the maximum and minimum of the thrust in the limits.

86. It is more usual to place the abutments as in Fig. 19. The space between them is usually covered with simple arches as ABCD. The horizontal thrusts

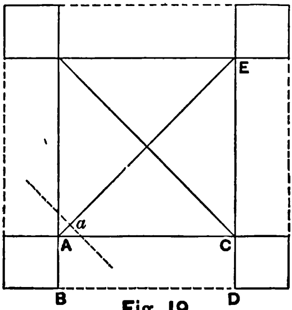


Fig. 19

of the two leaning against one abutment, acting at the crown joints are combined into one, $Q'\sqrt{2}$ acting directly over the

center of the abutment, and in the direction of a diagonal, as EA. The weights of the two semi-arches acting at their centers of gravity are combined into one, 2P′, acting at *a* on the diagonal AE. On drawing now a section of the abutment along AE produced and laying off the forces $Q'\sqrt{2}$, 2P′, T, P, H if any, and the weight of abutment, in their proper positions and combining these forces into one resultant, we ascertain if the center of pressure at the base of the abutment lies within proper limits: the middle third, or whatever limit is chosen from practical considerations. It will be found that the addition of the encompassing arches conduces to stability, the effect of the downward force 2P′, more than counteracting the effect of the force $Q'\sqrt{2}$.

87. The groined arch investigated in art. 81, *et seq.*, is considered by Scheffler; but the analytical solution proposed by him is too rough an approximation to be commended; and besides it errs, in part on the safe side, and in part otherwise,

so that it is not known whether the final result is on the safe side or not, especially as the line of pressures is made to touch the contour curves.

For definitions of groined and cloistered arches, domes, &c., the reader is referred to Mahan's Civil Engineering; which book likewise contains descriptions and drawings of several noted bridges.

CLOISTERED ARCHES.

88. In the cloistered arch, shown in plan, in Fig. 20, AB, BD, DC and CA, are straight lines, whilst EF is a simple arch

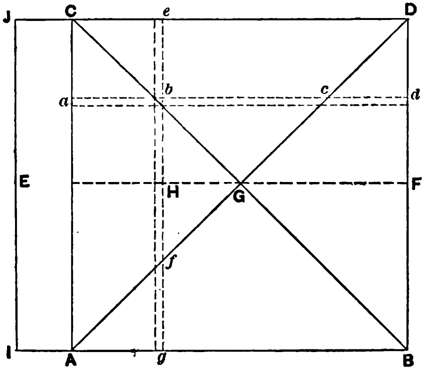


Fig. 20

of span EF, and rise equal to the height of the crown at G above the springing. AD and BC are the groins, forming the reentrant angles on which the smaller arcs, as *ab*, *be*, *cd*, *fg*, etc., meet with an inclined tangent. Thus *ab* is precisely similar in form to the part EH of the full center arch EF. The elements *bc* or *fb* are thus horizontal. Now the thrust at the crown G of the simple arch EF of small width, is horizontal, and is computed as for a simple arch. The arcs *ab* and *cd* sustain at *b* and *c* horizontal thrusts communicated through the horizontal element *bc*. When the centers are struck, the tendency to fall causes pressure on the voussoirs in four directions, $\perp$ and $\parallel$ AB, so that the elements, as *bc* and *bf* of the cylinders sustain a uniform horizontal compression in the directions *bc* and *bf*, and the voussoirs composing these elements sustain likewise an inclined thrust (except at the groins, where it is horizontal), in a direction perpendicular to the elements, whose amount is easily determined by the methods affecting simple arches.

89. If the above hypothesis be granted, it becomes an easy question to investigate the stability of a cloistered arch and its abutments, one of which is shown at AIJC.

Thus divide EG into any number of equal parts, and find by usual methods the weights and the positions of their centers of gravity, from the springing AC to any joint, in place of from G to the joint, as hitherto.

Part of the table made out then directly applies to each partial arc, as *ab*. On the elevation of the semi-arch EG and of each partial arch, as *ab*, pass curves of pressures, lying within the middle third, if possible; assuming the horizontal thrust at the top of the arch at the upper middle third limit as a first trial for the arc EF. With the tables made out as above, the resultant at the abutment must be combined, in turn, with the weights from the abutment to the joint considered, to find the centers of pressure on those joints.

We thus find the various horizontal

thrusts, acting at the groins CG and AG in a direction ⊥AC. On multiplying each of these thrusts by its vertical distance above the springing, and dividing the result by the sum of the thrusts, we find the vertical distance above the springing at which the resultant of the horizontal thrust T, of the part AGC acts. Similarly, find the horizontal distance to the resultant of the vertical forces, P acting on the part AGC; this resultant representing the weight of AGC. On combining these resultants, T and P acting in the vertical plane EG, with the weight of the abutment, as shown in Fig. 18, we ascertain whether the center of pressure on the base of the abutment falls within proper limits. Since the arc EF causes the greatest thrust, unless the abutment is made to act as one piece, as supposed above, its width should vary, being greatest at E and diminishing to nought, theoretically, at C; the intermediate widths being found in the usual manner from the thrusts of the partial arches resting

there. When the backing of the arches can resist a horizontal thrust, the curve of pressures for the smaller arches may be assumed (for safety in designing the abutment) to pass through the centers of the joints at the summit and abutment, or even lower at the summit and higher at the abutment; especially if the stones are not cut to fit snugly.

The writer has not met hitherto with a proper solution of either the groined or cloistered arch, and therefore commends the above to the attention of engineers.

DOMES.

90. The soffit of the dome will be supposed to be generated by revolving a curve about the vertical line representing the rise of the arch called the axis, so that every horizontal section of the soffit is a circle. The extrados may be generated by revolving a similar curve or any other figure about the axis. If we pass two meridian planes, making a small angle, ψ with each other, through the axis, we cut from the dome and backing,

if any, a solid FC, Fig. 21, being a part

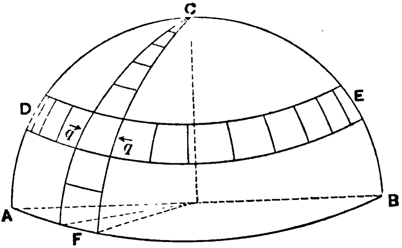


Fig. 21

of a wedge-shaped figure whose soffit is a *lune*. This solid, for want of a better name, we shall call a *lune solid.*

Now pass conical joints, perpendicular to the soffit, at certain distances apart; the part of the dome proper, as DE, lying between any two conical joints, will be called a *crown*.

91. We shall introduce the discussion by a quotation from Dr. Scheffler.*

"The authors who have treated the question of domes (NAVIER: *Résumé des leçons sur l'application de la mécanique,*

---

* "Theorie der Gewolbe;" also a French translation, "Theorie Des Voutes, &c."

part 1, No. 349; RONDELET: *Art. de bâtir*, liv. ix, sect. vi, chap. ii, etc.) have commonly divided the dome into lune solids (as defined above), and apply afterwards to these solids the same principles as to simple arches with vertical loads and horizontal thrusts. Now this view is entirely erroneous. It does not make known the influence of the forces which acts upon an arch of this character, and it implies this condition, impossible to realize; that the materials sustain at the summit a finite thrust upon an edge infinitely small.

"It is necessary, on the contrary, to divide the dome ABC, Fig. 21, into crowns, (as defined above), DE resting on the inclined bases of cones of revolution.

"It is proper first to examine the conditions of equilibrium of such a crown; which can moreover form the superior part of a dome open above.

"There are developed in these crowns horizontal pressures q, q, whose directions are normal to the joints of the

crown, and more intense in the upper than in the lower crowns.

"When we afterwards consider the lune solid CF, limited by meridian planes, it is necessary to combine the two forces q, q into a single horizontal force Q, acting outwards. It is necessary in all cases that the horizontal thrust at the upper joint may be null.

"This is evident for an open dome; for the dome closed at top, which is only a particular case of the open dome, it results from the fact, that the surface of the joint at the summit reduces to a line, which cannot support a finite pressure."

92. Let Fig. 22 represent a lune solid of the dome considered, and let P_1, P_2, P_3, P_4, laid off in order on the vertical line P_1 P_4, represent the weights of voussoirs 1, 2, 3, 4, respectively, with their loads. Let us assume, for the present, that the forces q, q, of the preceding figure act at the centers of the voussoirs; so that the forces Q_1, Q_2 . . , act through the centers of their corresponding voussoirs, 1, 2, . . . , and horizontally to the left in Fig. 22.

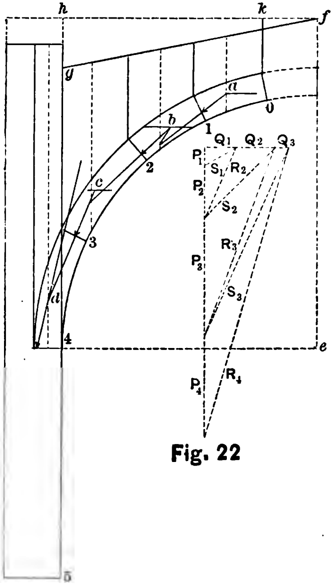


Fig. 22

Now the horizontal thrust at joint o is null. The weight P_1 of the first voussoir and load, acts through a and does not meet joint 1, so that there is no stability unless the "crown" including this voussoir, in tending to fall, exerts a horizontal pressure. The resultant Q_1 of the press-

ures exerted on both sides of voussoir 1 should be so great that when combined with P_1, the resultant shall cut the joint to which it refers, and make with the normal to this joint an angle not less than the angle of friction. These two conditions hold for every joint. If no joints open the resultants will lie in the middle third. Now if Q_1 be made so large that a line drawn through $a \parallel S_1$ the resultant of P_1 and Q_1, satisfies the above conditions, the point where it cuts joint 1 may be regarded as a *possible* center of pressure.

If the above conditions are not satisfied for an assumed value of Q_1, Q_1 must be increased.

Now extend the line through a, just drawn to intersection with the vertical through the center of gravity of the second voussoir and load, whose weight is P_2; from this point draw a parallel to R_2, the resultant of S_1 and P_2, and extend it upwards to intersection b, with the horizontal through the center of the second voussoir along which Q_2 acts. On draw-

ing through b a line to some point on joint 2; a parallel to it, in the force diagram, gives S_2, and cuts off Q_2, as shown in the figure. As before, if S_2 does not make an angle with the normal to joint 2 less than the angle of friction, Q_2 must be increased and the line through b made parallel to S_2 thus found. Similarly we proceed for other joints, until finally we get to a joint, as 3, below which no more forces of the type Q are needed to prevent the resultants on succeeding joints from falling *below* certain limits. The part of the lune solid below joint 3, called the "*joint of rupture*," thus acts as any simple arch; therefore we determine the resultant on joint 4 by combining S_3 and P_4—*i. e.*, by drawing through d a parallel to R_4 of the force diagram, the resultant on joint 4, to intersection with that joint.

Similarly, if we combine the resultant S_3 on joint 3 with the weight of the entire abutment, we find the centre of presssure on joint 5, which should lie within the middle third.

It will be observed that the resultants on joints 1, 2, 3, . . . , are represented in magnitude and direction by the lines S_1, S_2, . . . , of the force diagram, and in position by the arrow heads on the drawing of the arch.

The scale of the arch ring should be as large as can conveniently be drawn, since the lines determining the directions of the resultants are very short, and cannot be well shown on the small diagram. On that account, the preceding directions have been made full to conduce to clearness.

93. Scheffler now says, in effect, that if the voussoirs were absolutely incompressible, Q_1, Q_2, . . . should each in turn be the least that will cause stability and should therefore pass through the upper edges of joints 0, 1, 2, . . . ; and the resultants S_1, S_2, . . . should pass through the lower edges of joints 1, 2, etc., if the conditions affecting sliding are fulfilled. (On this point, see art. 106.)

But we know that actual voussoirs are compressible, so that if, as is usual, the

actual resultant on the springing joint passes to the left of the center, the outer edge is most compressed, and to allow this the haunches must spread and the top of the arch descend, so that about joint 3 the line of pressures passes below and at the top, above the center line. This is all the more evident if the springing joint opens at the inner edge. In the previous figure, the forces S_1, S_2, . . . , were drawn through the lower middle third limits. Now, if a dome acts like a stone bridge in a lowering of the crown, it would seem that the line of pressures there should lie above the center, so that its most probable position is at the upper middle third limit at the crown, and at the lower middle third limit at the joint of rupture, if the line of pressures can just be inscribed in the inner third of all the joints from the crown to the foundation of the abutment. When the line of pressures can be drawn within narrower limits, it is probably so confined actually.

An illustration of this view is given in article 105 following, which see.

It is plain that the forces q, q do not necessarily act at the centers of the voussoirs, as assumed. Their positions are indeterminate. Their least values for the same crown, consistent with no joint opening, corresponds to positions on the upper middle third limit, distant $\frac{1}{3}$ width voussoir from upper joint; their greatest values correspond to the lowest limiting positions.

As it is the object of the investigation to ascertain if the proper thickness of the arch ring and abutment have been chosen, it is well to err on the safe side in our hypotheses. In testing the stability of the abutment, it seems best to consider Q_1, Q_2, . . . , as acting at the centers of the voussoirs; the resultants S_1, S_2, . . . , at the centers of the joints. The latter hypothesis implies a uniform compression on the joints down to the joint of rupture; whereas, in fact, the line of pressures, as explained above, must pass below the center at this joint, giving generally a less total horizontal thrust.

As a modification of the above hypothesis, we may assume that the resultants S_1, S_2, . . . , are tangent to the center line, from the crown to the joint of rupture. It will be found that this involves raising Q_1, Q_2, . . , slightly above the centers of the voussoirs. The construction is much simplified by this assumption which will be illustrated more fully in Art. 100.

94. From the definitions of arts. 90 and 91, and a plan of a voussoir bounded by the two meridian planes whose included angle in arc is ψ, and which is solicited by the two horizontal forces q, q, (acting perpendicular to its vertical faces), whose resultant is Q, we have,

$$\tfrac{1}{2}Q = q \sin \tfrac{1}{2}\psi.$$

If a=half span, and ε=horizontal width of voussoir at the springing, then $a\psi = \varepsilon$. When the angle ψ is small, *i. e.*, when ε is made small enough, we have from the above equation,

$$q = \frac{a}{\varepsilon} Q;$$

from which q_1, q_2, . . can be computed, as soon as Q_1, Q_2, are found by the construction above.

95. *Numerical Example.*—Let us take the half span (Fig. 22) equal to 9.42, the depth of arch ring 0.94; and let the inclined line fg limit the load, the point f being 12.24 above the center e of the soffit, and g, 1.98 lower than f. The radius fk=1.86. Now divide the horizontal hk into six parts, each 1.26 wide, in place of three as before; drop verticals through the points of division, and from their intersection with the extrados draw the joints 0 to 6. We shall suppose approximately that the figures so formed are trapezoids, whose area equals the mean height multiplied by the width.

But now each trapezoidal solid, included between the two meridian planes, has a different thickness. Since the plan of the solid cut from the dome by the two meridian planes is a triangle, if we call its thickness at the middle of the springing joint 1, we find the thickness at the other mean verticals by multiplying 1 by the

ratio of the horizontal distance of the mean vertical from *ef* to the radius of the center of the springing joint about *ef*. By regarding each trapezoidal solid as having the thickness at its mean vertical, we introduce an error which diminishes indefinitely with ψ, and can thus be made as small as we please.

In the following table, column (1) refers to the joint, column (2) gives the height of the trapezoidal solid, column (3) its width, column (4) its thickness, and column (5) their product representing the forces P_1, P_2, ...

(1)	(2)	(3)	(4)		(5)
1	2.65	1.26	.25	P_1	0.83
2	2.83	1.26	.38	P_2	1.35
3	3.23	1.26	.50	P_3	2.03
4	3.92	1.26	.63	P_4	3.11
5	5.03	1.26	.76	P_5	4.82
6	7.05	1.26	.89	P_6	7.90
7	10.99	0.94	1.00	P_7	10.33

The volumes of the voussoirs and loads can be exactly determined by the principle of Guldinus (see Weisbach's "Mechanics," Coxe, Vol. 1, p. 126), that *the*

contents of a solid of rotation is equal to the product of the generating surface and the space described by its center of gravity while generating the body.

For greater accuracy the voussoirs and loads may be considered separately, and their common center of gravity and volume found by combining them afterwards.

The above principle we shall use in some subsequent constructions.

The construction is now proceeded with exactly as described for Fig. 22, which is in fact a drawing to these dimensions.

The induced forces Q_1, Q_2, . . . , were conceived to pass through the centers of the voussoirs. The resultants S_1 S_2, . . . , on the joints were made first to pass through the lower middle third limits, and afterwards through the center of those joints. In both cases the joint marked 3 in Fig. 22 was the joint of rupture: the resultant on the springing joint, in the first instance, coinciding with the resultant as drawn in Fig. 22;

in the last case passing nearly through the extrados. The total horizontal thrust in the first case$=7.75$; in the last, 8.13. If we take the width of abutment at 3, its height above the springing 10.99, its depth below it 7.01; its mean thickness is $\frac{10.92}{9.9}=1.1$, and its total volume, including a part of the arch is $3\times18\times1.1=59.4$; which combined with the resultant S_6 on the joint marked 3 in Fig. 22 cuts the base, for the first case noted above, only 0.05 outside of the middle third, in the last case 0.18 outside.

The force $Q_2=2.3$ is the largest of the forces Q_1, Q_2, . . . , whence by art. 94, $q=\frac{a}{e}\,Q_2=9.9\times2.3=22.77$ cubic units of stone. The force q acts on an area of 1.22 square units. There is evidently no danger of crushing from the horizontal thrust around the second crown from the top, as stone will bear on a square unit a pillar of a square unit section and several thousand units high. Similarly for the resultants on all the joints. We con-

clude that with the backing used, or by increasing the depth of arch ring at the springing about one-third, the arch will be stable. A very slight increase in the width of abutment will prevent any joint from opening.

In the preceding example any unit of length may be taken, as foot, meter, etc.

96. We took the thickness of voussoir at springing as unity. Since ψ should be a very small angle for greater accuracy in the assumptions about center of gravity and the values of q, etc., it may be thought that the thickness is too great. But since multiplying the weights of voussoirs and abutments by the same quantity does not change their ratios, the thickness taken is really immaterial except in finding q, since the centers of gravity are assumed to lie in the same verticals on the vertical projection of a medial meridian section, no matter what the thickness may be.

In truth, the assumption about the position of the center of gravity of a voussoir and load is more nearly realized when the thickness is appreciable.

97. When the soffit and exterior surface of the dome are both surfaces of spheres having the same center A (Fig. 23), the volumes of the voussoirs are

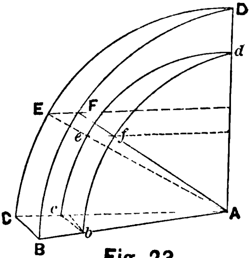


Fig. 23

easily obtained. Thus let $r = AD =$ radius of outer sphere; one half its volume is $\frac{2}{3}\pi r^3$. The wedge ABCD is $\frac{1}{n}$ of this volume, if BC is $\frac{1}{n}$ of the circumference $2\pi r$ $\therefore$ volume of wedge ABCD$=\frac{2\pi r^3}{3n}$.

Now divide the altitude AD into m equal parts (2 in the Fig.) and pass horizontal planes through the points of division; the surface of the lune BCD is

divided into m equal parts, by geometry, and the pyramids formed on these parts, as bases, with the center of the sphere as the common vertex, are therefore equal.

The volume of such a pyramid, as A—BCEF, is consequently, $\frac{2\pi r^3}{3mn}$.

Similarly the volume of a pyramid of the sphere, whose radius, $r' = \overline{Ad}$ is that of the soffit, and which is bounded by the same planes as any one of the preceding pyramids is, $\frac{2\pi r'^3}{3mn}$, since these pyramids, as *Abcef*, are diminished images of the preceding ones; every corresponding line being diminished in the ratio of r' to r. It follows that if horizontals are drawn through the ends f, . . . , of the edges of the second set of pyramids, they will divide the distance $\overline{Ad}$ into m equal parts. The same holds for similar pyramids of any sphere; so that if the center line of the arch ring on the sectional elevation, as *de* in Fig. 24, is divided into equal parts and horizontals be drawn through the points of divi-

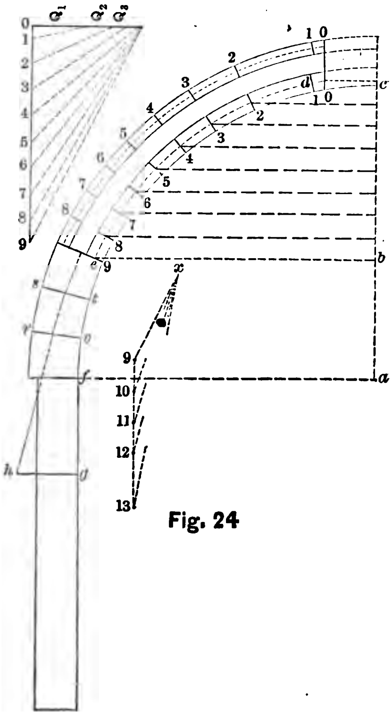


Fig. 24

sion to *de* and the joints be drawn through the latter points and the center of the sphere, the voussoirs so formed are all equal in volume.

Recurring to Fig. 23, we see that the volume of the voussoir Be=difference in volume of the two pyramids$=\frac{2\pi}{3mn}(r^3-r'^3)$.

98. In the open dome however, as Fig. 24, it is not always convenient to divide the rise ac into such equal parts that certain of the points of division will lie on the horizontals through d and e. We proceed then as follows: By geometry the area of the zone formed by revolving an arc as rs (Fig. 24) about the rise ac is equal to the altitude h of rs multiplied by $2\pi r$, r being the radius $\overline{as}$ of the sphere. Pass now two meridian planes through ac, whose included angle is $\frac{1}{n}$ of a circumference. The part of the zone included between them has an area $\frac{2\pi rh}{n}$; so that the pyramid formed on this base with a vertex at a, has a volume $\frac{2\pi r^2h}{3n}$.

Similarly the pyramid having the part

of the zone represented by *tv* as a base, has a volume, $\frac{2\pi r'^2 h'}{3n}$ where r'=radius *at* and h'=altitude of arc *tv*. Therefore the volume of the voussoir *rstv* included between the meridian planes and the conical joints *rv* and *st* is,

$$V=\frac{2\pi}{3n}(r^2h-r'^2h')$$

where r and h are the radius and altitude of the exterior arc, r' and h' of the interior. As before shown, if the altitudes of the type h are made equal in successive arcs, the values h' will all be equal. The divisions into equal altitudes can best be made along the center line in consequence of what follows.

99. Let us refer again to the inner dome *de* (Fig. 24).

If we pass horizontal planes midway between the horizontals drawn, also pass conical joints through their intersection with the center line *de*, we divide the previous voussoirs exactly in half, so that *the centers of gravity* of the first vous-

soirs lie on these supposed intermediate conical joints. They also lie nearly on the center line *de*, and the error of so regarding them can be made as small as we choose by sufficiently diminishing ψ, the angle included between the two meridian planes and the height of voussoir.

The centers of gravity of the voussoirs will therefore be assumed to be on the center line of the elevation of the medial meridian section *de*, at the intersections of the horizontals drawn midway between the first horizontals drawn that divide *bc* into equal parts.

100. Fig. 24 represents a meridian section of the Church of St. Peter, at Rome. The dimensions given by Scheffler, as I understand them, are as follows:

The radius of the soffit is 72 feet, and of the outer surface, $\overline{as} = 83.8$. At 31 feet above the springing, the structure is composed of two domes, the outer having a thickness of 2.6 feet, the inner being 4.1 thick at *d* and 5.1 at *e*, so that the center line *de* is described from a center slightly below *a* on the vertical

$\overline{ac}$ produced. The dome has an opening at top 12.4 radius, and the lantern supported at the top is equivalent in weight to a block of stone 2.1×56.6, of which the outer shell supports one-third and the inner two-thirds. The first voussoir at the top, in both shells, is made 2.1, horizontal width; the altitude of the center line, $\overline{bc}$, for the part *de*, is then divided into 8 equal parts and the joints drawn as in the figure, a similar construction applying to the outer shell. The part below the two shells is similarly divided into 3 equal parts. Applying the formula just deduced in Article 98, measuring the altitudes on a drawing to a scale of 4 feet to the inch, we find the volumes of the voussoirs like *rstv*, $\frac{2\pi}{3n}$ 28696, the voussoirs of the outer shell, $\frac{2\pi}{3n}$ 3957, except the top one, whose volume is $\frac{2\pi}{3n}$ 359. cubic feet. The volume of the top voussoir of the inner shell is $\frac{2\pi}{3n}$ 501 cubic feet. The voussoirs

2 to 9 of the inner shell, were each, in turn, assumed to have an outer surface concentric with the soffit, of radii equal to the mean radii of the outer surface for the voussoir considered, *i.e.*, equal to 72 + mean thickness in feet of voussoir. We thus find the volumes of voussoirs 2 to 9 equal to the constant $\frac{2\pi}{3n}$ multiplied in turn by 4695, 4805, 4915, 5000, 5124, 5206, 5267 and 5400.

The volume of the lantern, by the law of Guldinus (art. 95) $= 2.1 \times 56.6 \times \frac{2\pi \times 13.45}{n} = \frac{2\pi}{3n}$ 4794, one-third of which is added to the volume of voussoir 1 of outer shell and two thirds to that of voussoir 1 of the inner shell. The part *fgh*, 10.3 wide and 23.7 high, has a volume, $10.3 \times 23.7 \times \frac{2\pi \times 77.15}{n} = \frac{2\pi}{3n}$ 56500. This part has not the full width of the bottom voussoirs as drawn in the figure.

We now lay off on vertical lines, the weights just found, omitting the common constant $\frac{2\pi}{3n}$.

The loads affecting the outer shell are laid off to its left; those pertaining to the inner shell just below its center about (not shown in figure).

To be on the safe side, we shall assume that the resultants on the joints from the summit to the joint of rupture are tangent to the center line of the ring. Thus for the outer shell, draw through the points 1, 2, inches, of the force lines, parallel to tangents to the center line at joints 1, 2, . . . (or $\perp$ to radii). These lines cut off successive distances on the horizontal through *o*, equal to the radial forces Q_1, Q_2, . . . exerted by the successive crowns 1, 2, . . .

We find that below joint 6 there is no longer a radial force needed; so that below that joint the curve of pressures is continued to joint 9 as in a simple arch.

Similar results were found for the inner shell. The centers of pressure on joints 9 of the outer and inner shells are at the outer middle third limit for the outer shell, and slightly above the center line for the inner shell. This neces-

sitates spreading about joints 6, so that the line of pressures there is below the center line, so that the actual horizontal thrust is less than estimated, as stated above.

Now combining the resultants at joint 9 into one, laying off $\overline{x9}$ equal and parallel to it, its position being found by moments, we continue the line of pressures as per dotted line to joint *hg*. The successive volumes of voussoirs are laid off on the force line 9 . . . 13. This second force diagram is drawn to a smaller scale than the preceding.

101. At joint *hg* this line of pressures passes outside the joint so that the dome cannot be regarded as sufficiently stable in itself. If rotation occurs the line of pressures would approach the extrados at the summit, the intrados at the joints of rupture and the extrados at joint *hg*.

By encircling the dome just above the springing by a band of iron of sufficient cross section, stability may be assured. The band may be applied above the springing if desired. It evidently is

much less effective in preventing deformation of the dome if applied below the springing as was done in this case.

The total horizontal thrust of the lune solid (being $\frac{2\pi}{3n} \times$ the horizontal component of $\overline{x9}$) is $Q=39600 \frac{2\pi}{3n}=13200\psi$ cubic feet of stone$=924.\psi$tons, if we put the weight of a cubic foot of stone at 157 lbs.$=.07$ ton.

If this is to be entirely destroyed by the iron band, so that the resultants below the springing will all be vertical, we have the strain on the band by art. 94, when ψ is small

$$q=\frac{Q}{\psi}=924 \text{ tons.}$$

Now iron, exposed to a dead strain alone, may safely be subjected to a strain of $7\frac{1}{2}$ tons per square inch; so that the bar may have a cross section of 123 square inches.

If the iron stretches $\frac{1}{12000}$ of its length for every ton per square inch,

the ring whose diameter is 168 feet will elongate 0.33 feet, so that the diameter is increased 0.04 feet. There will consequently be a slight deformation of the arch, in consequence the top of the abutment moves slightly outwards, and the pressure on its base is not generally vertical; *i. e.*, the iron band has not totally destroyed the horizontal thrust. The action of the band is like that of a radial force acting inwards and equal, or nearly equal, to the total horizontal thrust.

If the hoop encircles the dome just below the springing of the two shells, it will prevent deformation of the arch also; for there can be no spreading outwards at this point, any tendency that way being met by the resistance of the hoop, which thus supplies sufficient horizontal force to force the line of pressures below it to keep within the joint areas a certain distance, dependent upon the spreading of the arch at the hoop. If this spreading is inappreciable, then the hoop exerts force enough to restrain the

line of pressures to the centers of the joints nearly below it.

There is therefore no necessity in the voussoir dome for additional hoops below the first, unless the first is unable to destroy the outward thrust. The problem is then really indeterminate of ascertaining the precise amounts of the strains sustained by the several hoops. The one nearest the joint of rupture of course will sustain by far the greatest part; the hoops, at joints where no spreading would occur, if they were not applied, not sustaining any. The resultants on the abutment joints cannot approach the outer limits without the top of abutment moving outwards; as this is prevented by the top hoop principally, it is evident that the one hoop should be placed not far below the joint of rupture. It would seem best to make these hoops of steel as it does not stretch as much as iron. Wire cables with a means of tightening would be especially convenient.

102. We see that the thrust of the type Q is greatest on voussoirs 1 of both shells. Thus for outer shell, Q_1

$$=16900\frac{2\pi}{3n}=5633\ \psi$$

Thus for inner shell, $Q_1=6000\ \psi$, which multiplied by .07 gives the thrusts in tons. By art. 94, we have $q=\frac{Q_1}{\psi}$ or 394 tons and 420 tons respectively. Now voussoir 1 of outer shell has an area of 6 square feet; the lower voussoir 1 an area of 10.45 square feet, so that the pressures per square foot are 66 and 40 tons respectively, which good stone can stand easily.

The thrusts Q_1 of both shells is probably less than assumed, for the force diagrams indicate a small value for Q_2—in fact for the lower shell Q_2 nearly vanishes—but the compression around the ring of the first crown would necessarily bring the second more in action thereby increasing its thrust. The tendency then is to equalize more nearly the values of the thrusts Q_1, Q_2, . . . than as given by our construction.

103. It is evident that the greatest economy is subserved, by the employ-

ment of one or more thin domes to a short distance below the joint of rupture, as is the practice generally in large domes. One shell would suffice if the weight of lantern (if any) could be carried by it; otherwise two or more should be used.

104. The abutment below joint *hg* is counterforted so as to present a greater width than shown in the figure. The introduction of the hoop of course prevents any movement in it, so that the stability of the whole structure is assured.

105. We shall give now the spherical dome closed at top to illustrate the view taken in art. 93 of the position of the actual line of pressures, besides other points not noticed before. This dome, Fig. 25, has a thickness of one fifteenth the span.

Divide the altitude of the center line of the ring into eight equal parts, draw horizontals, &c., as before. The lune solid is thus divided into eight equal parts which lay off on the force line 0 . . . 8.

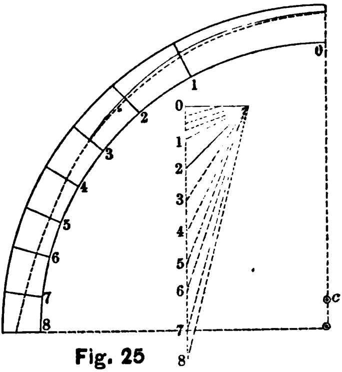


Fig. 25

Now a dome of this kind fails by rotating about the outer edges of joints at the crown and abutment, and the inner edges about joint 4; each lune solid separating from the others and acting as simple arches.

For a dome then of small stability the line of pressures passes nearly through the top, the intrados edge at joint 4 and the outer edge at joint 8.

For a dome of greater stability its position depends upon the amount of spreading at the haunches, and the consequent rocking at joint 8. If a line of

pressures can just be inscribed in certain limits, equally distant from the center, then it is probable, from a consideration of the way in which an arch settles, that the actual line of pressures touches these limiting curves towards the extrados side at the top and abutment, and next the intrados side at the joint of rupture.

Now if such a curve can be inscribed in the middle third no joints will open, and we may conclude that the dome has sufficient stability.

Draw an arc of a circle through the upper limit at the summit and the lower limit at joint 4 with a center *c* and assume that this arc coincides with the line of pressures a certain distance from the top, the resultants being tangent to it.

Then at some joint as 2 continue the pressure line down to the springing with the horizontal thrust found at 2. If the line so found does not keep within the middle third, let it be commenced at another joint, until one is found that will satisfy the conditions. On dividing

voussoir 2 into four others, it was found that a line of pressures, continued as for a simple arch, from where the arc above cuts the upper joint of voussoir $1\frac{3}{4}$, keeps almost entirely in the middle third, as shown by the dotted line: cutting joints 4 and 5 at the lower limits and joint 8 at the outer limit.

This is therefore a *probable* curve of pressures as determined from considerations of how an arch settles.

106. The construction given in art. 93, gives the lines of pressures, that beginning at the summit, first makes Q_1 a minimum, and then, for the same line of pressures in order, Q_2, Q_3, . . . *But this does not make, necessarily, the total thrust*, $Q_1 + Q_2 +$ . . . , *in the lower part of the arch a minimum.* In fact, this total thrust determined in this way, restricting the line to the inner third is found to be $\frac{1}{4}$ to $\frac{1}{6}$ greater than the thrust, determined as follows, that corresponds to the *minimum of the total horizontal thrust*, $Q_1 + Q_2 + \cdot$ . . . , within the limits taken; as first given, in effect,

by Prof. Eddy, in his "New Constructions in Graphical Statics." Take the upper middle third limit as the line of pressures down to a joint ($1\frac{1}{8}$ in this case), where the horizontal thrust may become constant, as for a simple arch, the pressure line below this point remaining within the inner third and just touching the inner limit at some point. The line so drawn coincides nearly with the first below joint 3. As mentioned in art. 93, the thrusts Q_1, Q_2, . . . will, by this construction, be really slightly outside of the inner third limits. The point where the simple arch begins is higher than in the previous cases. If the center line of the arch ring is assumed for the line of pressures, a certain distance from the summit the joint of rupture is lowered, *i.e.*, a less part of the lune solid acts as a simple arch, and the horizontal thrust is increased.

If the curve of pressures is taken to coincide with an arc starting at the upper limit at the crown as before, and lying below the previous arc, the thrusts Q_1,

Q_2, . . . near the crown are lessened; but the total horizontal thrust will be greater than as found in art. 105, as is evident.

Now, of all such arcs it is impossible to say which one, if any, is the true line of pressures, since this is a function of the deformation of the arch.

It may be remarked further that the direction of the thrusts near the summit are most likely more inclined than drawn above, since, by the construction above, a very small crown of voussoirs next the summit exerts a comparatively large thrust; so that the upper crown is compressed sufficiently to bring the next crown more in action, and so on down. If we divide voussoir 1 into four equal ones, we find that the circumferential thrusts *per square unit* on each crown going from the top are proportional to 2.5, 2.3, 1.9, and 1.8 respectively, so that the unit strains decrease going from the summit as stated. It would certainly give a large stability to apply the method of art. 100 and require that the line of pressures so drawn should keep within

the inner third below the joint of rupture. The arch cannot fall unless the line of pressures nearly touches the contour curves; hence, of the infinite number of positions it can take, it would seem that the first just drawn should offer sufficient stability, though it may not be exactly the true one.

107. If there is a weight at the summit, since its effect is to move the point *o* of the force line upwards and thus increase the thrusts in the top crowns, the line of pressures with a constant horizontal thrust must commence nearer the summit than before. The reverse happens when any weight is taken from the top of the dome, as for an open dome. In the former case, a small weight at top will cause nearly the whole of the lune solid to act as a simple arch.

CONICAL DOME.

108. Let Fig. 26 represent a meridian section of a conical dome. Divide the altitude $\overline{cd}$ into eight equal parts, and pass horizontal planes through the points

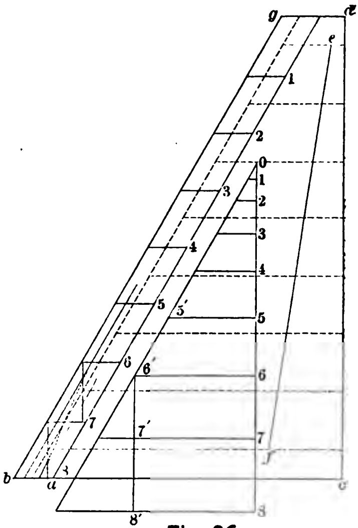


Fig. 26

of division giving the joints 1, 2, . . . Midway between these joints draw the horizontal radii of the center line shown by the dotted lines. If we call the horizontal thickness of the ring $ab = t$, the vertical distance between any two joints h, and the mean radius of the center line between these joints by r, we have for

the volume of the voussoir included between the two joints and two meridian planes, making an angle ψ with each other,

$$V = htr\psi,$$

according to the law of Guldinus.

The weights of the successive voussoirs vary therefore as r. Therefore lay off on the vertical force line, 0 . . . 8 successive distances, proportional to the dotted radii, beginning with the first voussoir. The line $\overline{ef}$ cuts off $\frac{1}{4}$ of these radii counting from $\overline{cd}$.

Next draw the line $\overline{06'} \parallel \overline{bg}$ and pass horizontals through the points 1, 2, . . . of the force line. If the resultants on the joints are assumed to coincide with, or be parallel to the center line, the hypothenuses of the triangles just formed represent the strains on the joints. Thus $\overline{06'}$ is proportional to the strain on joint 6, and $\overline{66'} - \overline{55'}$ = horizontal radial force Q exerted by the sixth ring.

Now if the arch is assumed practically undeformable, the center line is the line

of pressures, and the force diagram is sufficiently correct.

For compressible materials however, if the top and bottom are kept from moving horizontally, the middle of the dome from the compression of its rings tends to move inwards, which requires that the line of pressures there lies outside of the center line, and, as a consequence, inside the center line at top and bottom. Its exact position is indeterminate and is dependent upon the method used in building (whether with or without centers or supports) also on the fitting of the stones. If the abutments spread, as probably happens, the wedge-shaped solids, included between the meridian planes, tend to separate next the base; and the inner edges of joints next the abutment must bear the most, *i.e.* the line of pressures there is inside of the center line.

We see, therefore, that the assumption that the center line is the line of pressures, is on the side of safety, so that the force diagram above may be used in de-

termining the dimensions of abutments or hoops to withstand the horizontal thrust.

109. The least horizontal thrust, consistent with no joints opening, may be found as follows (also see Eddy's New Constructions, &c.):

Assume the centers of gravity of the voussoirs to lie on the center line midway between the joints, which hypothesis can be made as near the truth as we choose by sufficiently diminishing ψ. Now combine the thrust at joint 6, $\overline{06'}$ (as found above), supposed to act at the exterior middle third limit, with the weight of voussoirs below it; if the resultant, $\overline{08'}$, strikes the base at the inner limit, the horizontal thrust, $66' = 88'$, is a minimum in order that no joints open. The dome, however, may be perfectly stable when joints open, so that the smallest thrust consistent with stability is less than the above. In the above figure it was found, on a second trial, that at joint $6\frac{1}{4}$ the horizontal thrust first became constant, so that the part of the

dome below it exerted no circumferential thrust on the hypothesis. The point of contact with the outer limit is probably nearer the middle of the dome section, as the yielding is greatest there.

The two constructions given thus indicate limits between which the true thrust is found.

110. The preceding construction is the concluding one for this series.

In what has preceded we have been careful to state clearly the hypotheses which are introduced, and to criticise them in the light of both theory and facts.

It is not difficult for a mathematician, having made certain assumptions, to develop, perhaps, an elegant theory, quite dazzling to the inexperienced; but the engineer requires that the hypotheses be proved correct, or at least nearly so, before he can use them in practice.

So far from enforcing an unproved hypothesis by confident assertions, the writer may have leaned too much the other way by too often employing the

word "probable" where a stronger adjective would have better suited, especially in the parts referring to true curves of pressure. It is evident that the difficulty of locating the real pressure curve in a simple arch is increased for its combinations in a much greater ratio.

It has been pointed out, however, that the arch may satisfactorily be tested without its true position being known, from considerations of arches at the limit of stability. In fact, it is believed that this true position is closely approximated to by drawing a curve of pressures, within limits, approximately equidistant from the center line, corresponding to the maximum and minimum of the thrust (see art. 27).

This principle receives corroboration from certain constructions pertaining to the solid arch, which apply to the voussoir arch when the stones are cut perfectly, the mortar joints very thin, and no joints open, the spandrels not being supposed to exert any resistance.

In such cases, the pressure curve for the solid arch is identical with that for the voussoir arch.

This view will be clearly exposed in a work on "Solid Arches," in preparation.

**** Any book in this Catalogue sent free by mail, on receipt of price.*

VALUABLE
SCIENTIFIC BOOKS,
PUBLISHED BY
D. VAN NOSTRAND,
23 Murray Street, and 27 Warren Street,

NEW YORK.

WEISBACH. A MANUAL OF THEORETICAL MECHANICS. By Julius Weisbach, Ph. D. Translated by Eckley B. Coxe, A.M., M.E. 1100 pages and 902 wood-cut illustrations. 8vo, cloth, ... $10 00

FRANCIS. LOWELL HYDRAULIC EXPERIMENTS—being a Selection from Experiments on Hydraulic Motors, on the Flow of Water over Weirs, and in open Canals of Uniform Rectangular Section, made at Lowell, Mass. By J. B. Francis, Civil Engineer. Third edition, revised and enlarged, with 23 copper-plates, beautifully engraved, and about 100 new pages of text. 4to, cloth, ... 15 00

KIRKWOOD. ON THE FILTRATION OF RIVER WATERS, for the Supply of Cities, as practised in Europe. By James P. Kirkwood. Illustrated by 30 double-plate engravings. 4to, cloth 15 00

FANNING. A PRACTICAL TREATISE OF WATER SUPPLY ENGINEERING. Relating to the Hydrology, Hydrodynamics, and Practical Construction of Water-Works, in North America. With numerous Tables and 180 illustrations. By J. T. Fanning, C.E. 650 pages. 8vo, cloth extra, . $6 00

WHIPPLE. AN ELEMENTARY TREATISE ON BRIDGE BUILDING. By S. Whipple, C. E. New Edition. Illustrated. 8vo, cloth, 4 00

MERRILL. IRON TRUSS BRIDGES FOR RAILROADS. The Method of Calculating Strains in Trusses, with a careful comparison of the most prominent Trusses, in reference to economy in combination, etc., etc. By Bvt. Col. William E. Merrill, U. S. A., Corps of Engineers. Nine lithographed plates of illustrations. Third edition. 4to, cloth, ... 5 00

SHREVE. A TREATISE ON THE STRENGTH OF BRIDGES AND ROOFS. Comprising the determination of Algebraic formulas for Strains in Horizontal, Inclined or Rafter, Triangular, Bowstring Lenticular and other Trusses, from fixed and moving loads, with practical applications and examples, for the use of Students and Engineers By Samuel H. Shreve, A. M., Civil Engineer. Second edition, 87 woodcut illustrations. 8vo, cloth, . . . 5 00

KANSAS CITY BRIDGE. WITH AN ACCOUNT OF THE REGIMEN OF THE MISSOURI RIVER,— and a description of the Methods used for Founding in that River. By O. Chanute, Chief Engineer, and George Morison, Assistant Engineer. Illustrated with five lithographic views and twelve plates of plans. 4to, cloth, ... 6 00

CLARKE. DESCRIPTION OF THE IRON RAILWAY BRIDGE Across the Mississippi River at Quincy, Illinois. By Thomas Curtis Clarke, Chief Engineer. With twenty-one lithographed Plans. 4to, cloth, ... $7 50

ROEBLING. LONG AND SHORT SPAN RAILWAY BRIDGES. By John A. Roebling, C. E. With large copperplate engravings of plans and views. Imperial folio, cloth, . 25 00

DUBOIS. THE NEW METHOD OF GRAPHICAL STATICS. By A. J. Dubois, C. E., Ph. D. 60 illustrations. 8vo, cloth, ... 2 00

EDDY. NEW CONSTRUCTIONS IN GRAPHICAL STATICS. By Prof. Henry B. Eddy, C. E. Ph. D. Illustrated by ten engravings in text, and nine folding plates. 8vo, cloth, 1 50

BOW. A TREATISE ON BRACING—with its application to Bridges and other Structures of Wood or Iron. By Robert Henry Bow, C. E. 156 illustrations on stone. 8vo, cloth, 1 50

STONEY. THE THEORY OF STRAINS IN GIRDERS—and Similar Structures—with Observations on the Application of Theory to Practice, and Tables of Strength and other Properties of Materials. By Bindon B. Stoney, B. A. New and Revised Edition, with numerous illustrations. Royal 8vo, 664 pp., cloth, ... 12 50

HENRICI. SKELETON STRUCTURES, especially in their Application to the building of Steel and Iron Bridges. By Olaus Henrici. 8vo, cloth, ... 1 50

KING. LESSONS AND PRACTICAL NOTES ON STEAM. The Steam Engine, Propellers, &c., &c., for Young Engineers. By the late W. R. King, U. S. N., revised by Chief-Engineer J. W. King, U. S. Navy. 19th edition. 8vo, cloth, ... 2 00

AUCHINCLOSS. APPLICATION OF THE SLIDE VALVE and Link Motion to Stationary, Portable, Locomotive and Marine Engines. By William S Auchincloss. Designed as a hand-book for Mechanical Engineers. With 37 wood-cuts and 21 lithographic plates, with copper-plate engraving of the Travel Scale. Sixth edition. 8vo, cloth, $3 00

BURGH. MODERN MARINE ENGINEERING, applied to Paddle and Screw Propulsion. Consisting of 36 Colored Plates, 259 Practical Wood-cut Illustrations, and 403 pages of Descriptive Matter, the whole being an exposition of the present practice of the following firms: Messrs. J. Penn & Sons; Messrs. Maudslay, Sons & Field; Messrs. James Watt & Co.; Messrs. J. & G. Rennie. Messrs. R. Napier & Sons: Messrs J. & W. Dudgeon: Messrs. Ravenhill & Hodgson: Messrs Humphreys & Tenant; Mr J. T. Spencer, and Messrs. Forrester & Co. By N. P. Burgh. Engineer. One thick 4to vol., cloth, $25.00; half morocco, 30 00

BACON. A TREATISE ON THE RICHARD'S STEAM-ENGINE INDICATOR — with directions for its use. By Charles T. Porter. Revised, with notes and large additions as developed by American Practice; with an Appendix containing useful formulæ and rules for Engineers. By F. W. Bacon, M. E. Illustrated Second edition. 12mo. Cloth $1.00; morocco, ... 1 50

ISHERWOOD. ENGINEERING PRECEDENTS FOR STEAM MACHINERY. By B. F. Isherwood, Chief Engineer, U. S. Navy. With illustrations. Two vols. in one. 8vo, cloth, 2 50

STILLMAN. THE STEAM ENGINE INDICATOR —and the Improved Manometer Steam and Vacuum Gauges—their utility and application. By Paul Stillman. New edition. 12mo, cloth, ... 1 00

MacCORD. A PRACTICAL TREATISE ON THE SLIDE VALVE, BY ECCENTRICS—examining by methods the action of the Eccentric upon the Slide Valve, and explaining the practical processes of laying out the movements, adapting the valve for its various duties in the steam-engine. By C. W. Mac Cord, A. M., Professor of Mechanical Drawing, Stevens' Institute of Technology, Hoboken, N. J. Illustrated. 4to, cloth, ... $3 00

PORTER. A TREATISE ON THE RICHARDS' STEAM-ENGINE INDICATOR, and the Development and Application of Force in the Steam-Engine. By Charles T. Porter. Third edition, revised and enlarged. Illustrated. 8vo, cloth, . . . 3 50

McCULLOCH. A TREATISE ON THE MECHANICAL THEORY OF HEAT, AND ITS APPLICATIONS TO THE STEAM-ENGINE. By Prof. R. S. McCulloch, of the Washington and Lee University, Lexington, Va. 8vo, cloth, ... 3 50

VAN BUREN. INVESTIGATIONS OF FORMULAS—for the Strength of the Iron parts of Steam Machinery. By J. D. Van Buren, Jr., C. E. Illustrated. 8vo, cloth, . 2 00

STUART. HOW TO BECOME A SUCCESSFUL ENGINEER. Being Hints to Youths intending to adopt the Profession. By Bernard Stuart, Engineer. Sixth edition. 18mo, boards, ... 50

SHIELDS. A TREATISE ON ENGINEERING CONSTRUCTION. Embracing Discussions of the Principles involved, and Descriptions of the Material employed in Tunneling, Bridging, Canal and Road Building, etc., etc. By J. E. Shields, C. E. 12mo. cloth, 1 50

WEYRAUCH. STRENGTH AND CALCULATION OF DIMENSIONS OF IRON AND STEEL CONSTRUCTIONS. Translated from the German of J. J. Weyrauch, Ph. D., with four folding Plates. 12mo, cloth, . . . $1 00

STUART. THE NAVAL DRY DOCKS OF THE UNITED STATES. By Charles B. Stuart, Engineer in Chief, U. S. Navy. Twenty-four engravings on steel. Fourth edition. 4to, cloth, ... 6 00

COLLINS. THE PRIVATE BOOK OF USEFUL ALLOYS, and Memoranda for Goldsmiths, Jewellers, etc. By James E. Collins. 18mo, flexible cloth, ... 50

TUNNER. A TREATISE ON ROLL-TURNING FOR THE MANUFACTURE OF IRON. By Peter Tunner Translated by John B. Pearse. With numerous wood-cuts, 8vo, and a folio Atlas of 10 lithographed plates of Rolls, Measurements, &c. Cloth, . . 10 00

GRUNER. THE MANUFACTURE OF STEEL. By M. L. Gruner. Translated from the French, by Lenox Smith, A.M., E.M.; with an Appendix on the Bessemer Process in the United States, by the translator. Illustrated by lithographed drawings and wood-cuts. 8vo, cloth, ... 3 50

BARBA. THE USE OF STEEL IN CONSTRUCTION. Methods of Working, Applying, and Testing Plates and Bars. By J. Barba. Translated from the French, with a Preface by A. L. Holley, P.B. Illustrated. 12mo, cloth, 1 50

BELL. CHEMICAL PHENOMENA OF IRON SMELTING. An Experimental and Practical Examination of the Circumstances which Determine the Capacity of the Blast Furnace, the Temperature of the Air, and the Proper Condition of the Materials to be operated upon. By I. Lowthian Bell. 8vo, cloth, ... 6 00

WARD. STEAM FOR THE MILLION. A Popular Treatise on Steam and its Application to the Useful Arts, especially to Navigation. By J. H. Ward, Commander U. S. Navy. 8vo, cloth, ... $1 00

CLARK. A MANUAL OF RULES, TABLES AND DATA FOR MECHANICAL ENGINEERS. Based on the most recent investigations. By Dan. Kinnear Clark. Illustrated with numerous diagrams. 1012 pages. 8vo. Cloth, $7 50; half morocco, ... 10 00

JOYNSON. THE METALS USED IN CONSTRUCTION: Iron, Steel, Bessemer Metals, etc, By F. H. Joynson. Illustrated. 12mo, cloth, ... 75

DODD. DICTIONARY OF MANUFACTURES, MINING, MACHINERY, AND THE INDUSTRIAL ARTS. By George Dodd. 12mo, cloth, 1 50

VON COTTA. TREATISE ON ORE DEPOSITS. By Bernhard Von Cotta, Freiburg, Saxony. Translated from the second German ed., by Frederick Prime, Jr., and revised by the author. With numerous illustrations. 8vo, cloth, ... 4 00

PLATTNER. MANUAL OF QUALITATIVE AND QUANTITATIVE ANALYSIS WITH THE BLOW-PIPE. From the last German edition. Revised and enlarged. By Prof. Th. Richter, o the Royal Saxon Mining Academy. Translated by Professor H. B. Cornwall. With eighty-seven wood-cuts and lithographic plate. Third edition, revised. 568 pp. 8vo, cloth, ... 5 00

PLYMPTON. THE BLOW-PIPE: A Guide to its Use in the Determination of Salts and Minerals. Compiled from various sources, by George W. Plympton, C. E., A. M., Professor of Physical Science in the Polytechnic Institute, Brooklyn, N. Y. 12mo, cloth, 1 50

JANNETTAZ. A GUIDE TO THE DETERMINATION OF ROCKS; being an Introduction to Lithology. By Edward Jannettaz, Docteur des Sciences. Translated from the French by G. W. Plympton, Professor of Physical Science at Brooklyn Polytechnic Institute. 12mo, cloth, ... $1 50

MOTT. A PRACTICAL TREATISE ON CHEMISTRY (Qualitative and Quantitative Analysis), Stoichiometry, Blowpipe Analysis, Mineralogy, Assaying, Pharmaceutical Preparations Human Secretions, Specific Gravities, Weights and Measures, etc., etc., etc. By Henry A. Mott, Jr., E. M., Ph. D. 650 pp. 8vo, cloth, ... 6 00

PYNCHON. INTRODUCTION TO CHEMICAL PHYSICS; Designed for the Use of Academies, Colleges, and High Schools. Illustrated with numerous engravings, and containing copious experiments, with directions for preparing them. By Thomas Ruggles Pynchon, D. D., M. A., President of Trinity College, Hartford. New edition, revised and enlarged. Crown 8vo, cloth, . . . 3 00

PRESCOTT. CHEMICAL EXAMINATION OF ALCOHOLIC LIQUORS. A Manual of the Constituents of the Distilled Spirits and Fermented Liquors of Commerce, and their Qualitative and Quantitative Determinations. By Alb. B. Prescott, Prof. of Chemistry, University of Michigan. 12mo, cloth, . 1 50

ELIOT AND STORER. A COMPENDIOUS MANUAL OF QUALITATIVE CHEMICAL ANALYSIS. By Charles W. Eliot and Frank H. Storer. Revised, with the co-operation of the Authors, by William Ripley Nichols, Professor of Chemistry in the Massachusetts Institute of Technology. New edition, revised. Illustrated. 12mo, cloth, ... 1 50

NAQUET. LEGAL CHEMISTRY. A Guide to the Detection of Poisons, Falsification of Writings, Adulteration of Alimentary and Pharmaceutical Substances; Analysis of Ashes, and Examination of Hair, Coins, Fire-arms and Stains, as Applied to Chemical Jurisprudence. For the Use of Chemists, Physicians, Lawyers, Pharmacists, and Experts. Translated, with additions, including a List of Books and Memoirs on Toxicology, etc., from the French of A. Naquet, by J. P. Battershall, Ph. D.; with a Preface by C. F. Chandler, Ph. D., M. D., LL. D. Illustrated. 12mo, cloth, ... $2 00

PRESCOTT. OUTLINES OF PROXIMATE ORGANIC ANALYSIS for the Identification, Separation, and Quantitative Determination of the more commonly occurring Organic Compounds. By Albert B. Prescott, Professor of Chemistry, University of Michigan. 12mo, cloth, ... 1 75

DOUGLAS AND PRESCOTT. QUALITATIVE CHEMICAL ANALYSIS. A Guide in the Practical Study of Chemistry, and in the work of Analysis. By S. H. Douglas and A. B. Prescott; Professors in the University of Michigan. Second edition, revised. 8vo, cloth, ... 3 50

RAMMELSBERG. GUIDE TO A COURSE OF QUANTITATIVE CHEMICAL ANALYSIS, ESPECIALLY OF MINERALS AND FURNACE PRODUCTS. Illustrated by Examples. By C. F. Rammelsberg. Translated by J. Towler, M. D. 8vo, cloth, ... 2 25

BEILSTEIN. AN INTRODUCTION TO QUALITATIVE CHEMICAL ANALYSIS. By F. Beilstein. Third edition. Translated by I. J. Osbun. 12mo. cloth, ... 75

POPE. A Hand-book for Electricians and Operators. By Frank L. Pope. Ninth edition. Revised and enlarged, and fully illustrated. 8vo, cloth, ... 2 00

SABINE. HISTORY AND PROGRESS OF THE ELECTRIC TELEGRAPH, with Descriptions of some of the Apparatus. By Robert Sabine, C. E. Second edition. 12mo, cloth, . . . $1 25

DAVIS AND RAE. HAND BOOK OF ELECTRICAL DIAGRAMS AND CONNECTIONS. By Charles H. Davis and Frank B. Rae. Illustrated with 32 full-page illustrations. Second edition. Oblong 8vo, cloth extra, . . . 2 00

HASKINS. THE GALVANOMETER, AND ITS USES. A Manual for Electricians and Students. By C. H. Haskins. Illustrated. Pocket form, morocco, ... 1 50

LARRABEE. CIPHER AND SECRET LETTER AND TELEGRAPAIC CODE, with Hogg's Improvements. By C. S. Larrabee. 18mo, flexible cloth, ... 1 00

GILLMORE PRACTICAL TREATISE ON LIMES, HYDRAULIC CEMENT, AND MORTARS. By Q. A. Gillmore, Lt.-Col. U. S. Engineers, Brevet Major-General U. S. Army. Fifth edition, revised and enlarged. 8vo, cloth, 4 00

GILLMORE. COIGNET BETON AND OTHER ARTIFICIAL STONE. By Q. A. Gillmore, Lt. Col. U. S. Engineers, Brevet Major-General U. S. Army. Nine plates, views, etc. 8vo, cloth, ... 2 50

GILLMORE. A PRACTICAL TREATISE ON THE CONSTRUCTION OF ROADS, STREETS, AND PAVEMENTS. By Q. A. Gillmore, Lt.-Col. U. S. Engineers, Brevet Major-General U. S. Army. Seventy illustrations. 12mo, clo., 2 00

GILLMORE. REPORT ON STRENGTH OF THE BUILDING STONES IN THE UNITED STATES, etc. 8vo, cloth, ... 1 00

HOLLEY. AMERICAN AND EUROPEAN RAILWAY PRACTICE, in the Economical Generation of Steam. By Alexander L. Holley. B. P. With 77 lithographed plates. Folio, cloth, 12 00

HAMILTON. Useful Information for Railway Men. Compiled by W. G. Hamilton, Engineer. Seventh edition, revised and enlarged. 577 pages. Pocket form, morocco, gilt, ... , . $2 00

STUART. The Civil and Military Engineers of America. By General Charles B. Stuart, Author of "Naval Dry Docks of the United States," etc., etc. With nine finely-executed Portraits on steel, of eminent Engineers, and illustrated by Engravings of some of the most important and original works constructed in America. 8vo, cloth, ... 5 00

ERNST. A Manual of Practical Military Engineering. Prepared for the use of the Cadets of the U. S. Military Academy, and for Engineer Troops. By Capt. O. H. Ernst, Corps of Engineers, Instructor in Practical Military Engineering, U. S. Military Academy. 193 wood-cuts and 3 lithographed plates. 12mo, cloth, . . 5 00

SIMMS. A Treatise on the Principles and Practice of Levelling, showing its application to purposes of Railway Engineering and the Construction of Roads, etc. By Frederick W. Simms, C. E. From the fifth London edition, revised and corrected, with the addition of Mr. Law's Practical Examples for Setting-out Railway Curves. Illustrated with three lithographic plates, and numerous wood-cuts. 8vo, cloth, ... 2 50

JEFFERS. Nautical Surveying. By William N. Jeffers, Captain U. S. Navy. Illustrated with 9 copperplates, and 31 wood-cut illustrations. 8vo, cloth, ... 5 00

THE PLANE TABLE. Its Uses in Topographical Surveying. From the papers of the U. S. Coast Survey. 8vo, cloth, . . 2 00

A TEXT-BOOK ON SURVEYING, PROJECTIONS, AND PORTABLE INSTRUMENTS, for the use of the Cadet Midshipmen, at the U. S. Naval Academy. 9 lithographed plates, and several wood-cuts. 8vo, cloth, . . $2 00

CHAUVENET. NEW METHOD OF CORRECTING LUNAR DISTANCES. By Wm. Chauvenet, LL.D. 8vo, cloth, ... 2 00

BURT. KEY TO THE SOLAR COMPASS, and Surveyor's Companion; comprising all the Rules necessary for use in the Field. By W. A. Burt, U. S. Deputy Surveyor. Second edition. Pocket-book form, tuck, . 2 50

HOWARD. EARTHWORK MENSURATION ON THE BASIS OF THE PRISMOIDAL FORMULÆ. Containing simple and labor-saving method of obtaining Prismoidal Contents directly from End Areas. Illustrated by Examples, and accompanied by Plain Rules for practical uses. By Conway R. Howard, Civil Engineer, Richmond, Va. Illustrated. 8vo, cloth, ... 1 50

MORRIS. EASY RULES FOR THE MEASUREMENT OF EARTHWORKS, by means of the Prismoidal Formulæ. By Elwood Morris, Civil Engineer. 78 illustrations. 8vo, cloth, 1 50

CLEVENGER. A TREATISE ON THE METHOD OF GOVERNMENT SURVEYING, as prescribed by the U. S. Congress and Commissioner of the General Land Office. With complete Mathematical, Astronomical, and Practical Instructions for the use of the U. S. Surveyors in the Field. By S. V. Clevenger, U. S. Deputy Surveyor. Illustrated. Pocket form, morocco, gilt, . . . 2 50

HEWSON. PRINCIPLES AND PRACTICE OF EMBANKING LANDS from River Floods, as applied to the Levees of the Missisipi. By William Hewson, Civil Engineer. 8vo, cloth, ... 2 00

MINIFIE. A TEXT-BOOK OF GEOMETRICAL DRAWING, for the use of Mechanics and Schools. With Illustrations for Drawing Plans, Elevations of Buildings and Machinery. With over 200 diagrams on steel. By William Minifie, Architect. Ninth edition. Royal 8vo, cloth, ... $4 00

MINIFIE. GEOMETRICAL DRAWING. Abridged from the octavo edition, for the use of Schools. Illustrated with 48 steel plates. New edition, enlarged. 12mo, cloth, 2 00

FREE HAND DRAWING. A GUIDE TO ORNAMENTAL, Figure, and Landscape Drawing. By an Art Student. Profusely illustrated. 18mo, boards, ... 50

AXON. THE MECHANIC'S FRIEND. A Collection of Receipts and Practical Suggestions, relating to Aquaria—Bronzing—Cements—Drawing—Dyes—Electricity—Gilding—Glass-working—Glues—Horology—Lacquers—Locomotives—Magnetism—Metal-working—Modelling—Photography—Pyrotechny—Railways—Solders—Steam-Engine—Telegraphy—Taxidermy—Varnishes—Waterproofing–and Miscellaneous Tools, Instruments, Machines, and Processes connected with the Chemical and Mechanical Arts. By William E. Axon, M.R.S.L. 12mo, cloth. 300 illustrations, ... 1 50

HARRISON. MECHANICS' TOOL BOOK, with Practical Rules and Suggestions, for the use of Machinists, Iron Workers, and others. By W. B. Harrison. 44 illustrations. 12mo, cloth 1 50

JOYNSON. THE MECHANIC'S AND STUDENT'S GUIDE in the designing and Construction of General Machine Gearing. Edited by Francis H. Joynson. With 18 folded plates. 8vo, cloth 2 00

RANDALL. QUARTZ OPERATOR'S HAND-BOOK. By P. M. Randall. New Edition. Revised and Enlarged. Fully illustrated. 12mo, cloth, ... $2 00

SILVERSMITH. A PRACTICAL HAND-BOOK FOR MINERS, METALLURGISTS, and Assayers. By Julius Silversmith. Fourth Edition. Illustrated. 12mo, cloth, ... 3 00

BARNES. SUBMARINE WARFARE, DEFENSIVE AND OFFENSIVE. Descriptions of the various forms of Torpedoes, Submarine Batteries and Torpedo Boats actually used in War. Methods of Ignition by Machinery, Contact Fuzes, and Electricity, and a full account of experiments made to determine the Explosive Force of Gunpowder under Water. Also a discussion of the Offensive Torpedo system; its effect upon Iron-clad Ship systems, and influence upon future Naval Wars. By Lieut.-Com. John S. Barnes, U. S. N. With 20 lithographic plates and many wood-cuts. 8vo, cloth, 5 00

FOSTER. SUBMARINE BLASTING, in Boston Harbor, Mass. Removal of Tower and Corwin Rocks. By John G. Foster, U. S. Eng. and Bvt. Major General U. S. Army. With seven Plates. 4to, cloth, 3 50

MOWBRAY. TRI-NITRO-GLYCERINE, as applied in the Hoosac Tunnel, and to Submarine Blasting, Torpedoes, Quarrying, etc. Illustrated. 8vo, cloth, . . . 3 00

WILLIAMSON. ON THE USE OF THE BAROMETER ON SURVEYS AND RECONNAISSANCES. Part I.–Meteorology in its Connection with Hypsometry. Part II.—Barometric Hypsometry. By R. S. Williamson, Bvt. Lt.-Col. U.S.A., Major Corps of Engineers. With illustrative tables and engravings. 4to, cloth, ... 15 00

WILLIAMSON. PRACTICAL TABLES IN METEOROLOGY AND HYPSOMETRY, in connection with the use of the Barometer By Col. R. S. Williamson, U. S. A. 4to, flexible cloth, $2 50

BUTLER. PROJECTILES AND RIFLED CANNON A Critical Discussion of the Principal Systems of Rifling and Projectiles, with Practical Suggestions for their Improvement. By Capt. John S. Butler, Ordnance Corps, U. S. A. 36 Plates. 4to, cloth, . . . 7 50

BENET. ELECTRO-BALLISTIC MACHINES, and the Schultz Chronoscope. By Lt.-Col S. V Benet, Chief of Ordnance U. S. A. Second edition, illustrated. 4to, cloth, . 3 00

MICHAELIS. THE LE BOULENGE CHRONOGRAPH. With three lithographed folding plates of illustrations. By Bvt. Captian O. E. Michaelis, Ordnance Corpse, U. S. A. 4to, cloth, ... 3 00

NUGENT. TREATISE ON OPTICS; or Light and Sight, theoretically and practically treated; with the application to Fine Art and Industrial Pursuits. By E. Nugent. With 103 illustrations. 12mo, cloth, . . . 1 50

PEIRCE. SYSTEM OF ANALYTIC MECHANICS. By Benjamin Peirce, Professor of Astronomy and Mathematics in Harvard University. 4to- cloth, ... 10 00

CRAIG. WEIGHTS AND MEASURES. An Account of the Decimal System, with Tables of Conversion for Commercial and Scientific Uses. By B. F. Craig, M. D. Square 32mo, limp cloth, ... , . . 50

ALEXANDER. UNIVERSAL DICTIONARY OF WEIGHTS AND MEASURES, Ancient and Modern, reduced to the standards of the United States of America. By J. H. Alexander. New edition. 8vo, cloth, . . 3 50